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Inverse Operators, *q*-Fractional Integrals, and *q*-Bernoulli Polynomials

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We introduce operators of q-fractional integration through inverses of the Askey–Wilson operator and use them to introduce a q-fractional calculus. We establish the semigroup property for fractional integrals and fractional derivatives. We study properties of the kernel of q-fractional integral and show how they give rise to a q-analogue of Bernoulli polynomials, which are now polynomials of two variables, x and y. As $q \to 1$ the polynomials become polynomials in $x-y$, a convolution kernel in one variable. We also evaluate explicitly a related kernel of a right inverse of the Askey–Wilson operator on an L^2 space weighted by the weight function of the Askey–Wilson polynomials.

Key Words: q-Fourier series; q-fractional calculus; q-Bernoulli polynomials; continuous q-ultraspherical polynomials; inverse of an Askey–Wilson operator; Askey–Wilson polynomials; q-Lommel polynomials.

1. INTRODUCTION

Zygmund [26] treats Weyl's approach to fractional integrals in Sections XII.8 and XII.9. He defines a fractional integral of order α of a function f which is integrable and of period 2π to be $I_\alpha f$,

$$(I_\alpha f)(x) \sim \sum_{n \neq 0} c_n \frac{e^{inx}}{(in)^\alpha}, \qquad \text{if} \quad f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \tag{1.1}$$

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with $i^\alpha = e^{i\pi\alpha/2}$. He then points out that

$$(I_\alpha f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \Psi_\alpha(x-t)\, dt, \qquad \Psi_\alpha(x) = \sum_{n \neq 0} \frac{e^{inx}}{(in)^\alpha}. \tag{1.2}$$

Zygmund [26] uses the notations $I_\alpha f$ and f_α interchangeably. He notes that the semigroup property $I_\alpha I_\beta = I_{\alpha+\beta}$ follows immediately from (1.1) for $\alpha, \beta > 0$. Of course I_1 coincides with $\int_0^x f(t)\, dt$ where $f \sim \sum f_n e^{inx}$ with $f_0 = 0$. Furthermore if $\alpha > n$ then $(d^n/dx^n)\, I_\alpha = I_{\alpha-n}$. When m is a positive integer $\Psi_m(x)$ is a constant multiple of the Bernoulli polynomial $B_m(x)$.

The purpose of this paper is to define a q-analogue of the fractional integral operators I_α, so that for $\alpha = 1$ it becomes a right inverse of the Askey–Wilson operator $\mathscr{D}_q$. The operator $\mathscr{D}_q$ will be defined below. Other one sided inverses on certain L^2 spaces have been introduced and studied in [8, 18, 17], and more recently in our work [16]. In order to describe our results we first remind the reader of the definitions of the Askey–Wilson operator and q-Fourier series.

Given a function $f(x)$ with $x = \cos\theta$, $f(x)$ can be viewed as a function of $e^{i\theta}$. Let

$$\breve{f}(e^{i\theta}) := f(x), \qquad x = \cos\theta. \tag{1.3}$$

The Askey–Wilson divided difference operator $\mathscr{D}_q$ [7] is defined by

$$(\mathscr{D}_q f)(x) = (\mathscr{D}_{q,x} f)(x) := \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{\breve{e}(q^{1/2}e^{i\theta}) - \breve{e}(q^{-1/2}e^{i\theta})}, \tag{1.4}$$

where $e(x) = x$. It follows from (1.4) that

$$(\mathscr{D}_q f)(x) = \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{i(q^{1/2} - q^{-1/2}) \sin\theta}. \tag{1.5}$$

The operator $\mathscr{D}_q$ was introduced in [7] and is a q-analogue of the differentiation operator. In fact $\mathscr{D}_q$ maps polynomials to polynomials, since

$$\mathscr{D}_q T_n(x) = \frac{1-q^n}{1-q}\, q^{(1-n)/2} U_{n-1}(x), \tag{1.6}$$

where T_n and U_n are the Chebyshev polynomials of the first and second kinds, respectively. A q-constant is a function annihilated by the Askey–Wilson operator. A q-polynomial is a function of the form $\sum_{j=0}^n a_j x^j$, where $a_0, ..., a_n$ are q-constants. These concepts were introduced in [15].

Note that (1.5) indicates that $\mathcal{D}_q$ remains invariant if q is replaced by $1/q$. In this work we will always assume $q \in (0, 1)$ and we shall follow closely the notation and terminology of basic hypergeometric series as in the books by Andrews *et al.* [4] and by Gasper and Rahman [12]. In particular

$$h(\cos\theta; a_1, ..., a_n) := \prod_{j=1}^{n} (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty.$$

The concept of a q-Fourier series first originated in a paper by Askey *et al.* [6]; see also [20]. But it was introduced in a more formal way by Bustoz and Suslov in [9]. Later, Suslov [22] introduced a version of q-Bernoulli polynomials, which will turn out to be different from what comes naturally from q-Fourier series; see Sections 3 and 4. The q-exponential function

$$\mathcal{E}_q(\cos\theta, \cos\phi; \alpha) := \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} (-e^{i(\phi+\theta)} q^{(1-n)/2}, -e^{i(\phi-\theta)} q^{(1-n)/2}; q)_n \times \frac{(\alpha e^{-i\phi})^n}{(q; q)_n} q^{n^2/4} \tag{1.7}$$

[18] plays a crucial role in q-Fourier series. It is straightforward to see that

$$\mathcal{D}_q \mathcal{E}_q(\cos\theta, \cos\phi; \alpha) = \frac{2\alpha q^{1/4}}{1-q} \mathcal{E}_q(\cos\theta, \cos\phi; \alpha). \tag{1.8}$$

If we let

$$\mathcal{E}_q(x; \alpha) = \mathcal{E}_q(x, 0; \alpha), \tag{1.9}$$

then $\mathcal{E}_q(0; \alpha) = 1$, and $\lim_{q\to 1} \mathcal{E}_q(x; (1-q)\alpha) = \exp(2\alpha x)$. The notation for $\mathcal{E}_q$ adopted here is the same as the one proposed by Suslov in [21] and is different from the original notation in [18].

In [18] where Ismail and Zhang introduced the $\mathcal{E}_q$ function, they also introduced q-analogues of the sine and cosine functions and used transformation formulas to analytically continue them to entire functions in the variable α. Bustoz and Suslov [9] identified a special case which leads to a complete orthogonal system of functions. This opened the door for a comprehensive study of q-Fourier series, where q-analogues of some results in classical Fourier series have been proven [9, 22], but many more questions remain open and deserve further investigation. Ismail [14] gave simple proofs of both the orthogonality and completeness of the q-exponential Fourier system and pointed out that this is just one example of a whole family of functions defined by series similar to the q-plane wave formula (1.14) given below.

The continuous q-ultraspherical polynomials play an important role in the q-Fourier analysis. They are generated by [5]

$$C_0(x; \beta \mid q) = 1, \qquad C_1(x; \beta \mid q) = 2x(1-\beta)/(1-q), \tag{1.10}$$

$$\begin{aligned} 2x(1-\beta q^n)\, C_n(x; \beta \mid q) &= (1-q^{n+1})\, C_{n+1}(x; \beta \mid q) \\ &\quad + (1-\beta^2 q^{n-1})\, C_{n-1}(x; \beta \mid q), \qquad n > 0. \end{aligned} \tag{1.11}$$

Their orthogonality relation is [5; 12, (7.4.15)]

$$\begin{aligned} &\int_{-1}^{1} C_m(x; \beta \mid q)\, C_n(x; \beta \mid q)\, w(x; \beta \mid q)\, dx \\ &\quad = \frac{2\pi(\beta, q\beta; q)_\infty}{(q, \beta^2; q)_\infty} \frac{(1-\beta)(\beta^2; q)_n}{(1-\beta q^n)(q; q)_n} \delta_{m,n}, \end{aligned} \tag{1.12}$$

where w is a weight function defined by

$$w(\cos\theta; \beta \mid q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\sin\theta (\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}, \qquad 0 < \theta < \pi. \tag{1.13}$$

The q-plane wave expansion mentioned earlier is [18]

$$\begin{aligned} \mathcal{E}_q(x; i\alpha/2) = {} & \frac{(2/\alpha)^\nu (q; q)_\infty}{(-q\alpha^2/4; q^2)_\infty (q^{\nu+1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(1-q^{n+\nu})}{(1-q^\nu)} q^{n^2/4} i^n \\ & \times J_{\nu+n}^{(2)}(\alpha; q)\, C_n(x; q^\nu \mid q), \end{aligned} \tag{1.14}$$

where

$$J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{(q, q^{\nu+1}; q)_n} q^{n(\nu+n)} \tag{1.15}$$

is one of Jackson's three q-Bessel functions [13]; see also Gasper and Rahman [12]. Formula (1.14) turned out to be very useful in deriving many results in this area, including addition theorems for $\mathcal{E}_q$, evaluation of integrals, and mathematical physics models [11].

In Section 2 we give a brief description of the q-Lommel polynomials $\{h_{n,\nu}(x;q)\}$, [13], which have the orthogonality property

$$\sum_{k\neq 0}^{\infty} \frac{A_{|k|}(\nu+1)}{j_{\nu,k}^2(q)} h_{n,\nu+1}(1/j_{\nu,k}(q);q)\, h_{m,\nu+1}(1/j_{\nu,k}(q);q) = \frac{q^{\nu n+n(n+1)/2}}{1-q^{n+\nu+1}}\,\delta_{m,n}, \tag{1.16}$$

where $j_{\nu,k}$ is the kth positive zero of the q-Bessel function $J_\nu^{(2)}(z;q)$ and $A_n(\nu+1)$ is related to its derivative at those zeros. In Section 3 we first introduce a formal definition of $\Psi_\alpha(x,y\,|\,q)$ and use (1.16) to derive the following convenient expression

$$\begin{aligned} \frac{2^\alpha q^{\alpha/4}}{(1-q)^\alpha}\Psi_\alpha(x,y\,|\,q) &= \frac{e^{-i\pi\alpha/2}}{4(1-\sqrt{q})}\sum_{r,s=0}^{\infty}\sum_{n\neq 0}\frac{A_{|n|}(3/2)\,i^{r-s}}{\omega_n^{2+\alpha}}\frac{(1-q^{r+3/2})(1-q^{s+3/2})}{q^{(r^2+s^2+2r+2s)/4}} \\ &\quad\times h_{r,3/2}(1/(2\omega_n);q)\,h_{s,3/2}(1/(2\omega_n);q) \\ &\quad\times C_{r+1}(x;q^{1/2}\,|\,q)\,C_{s+1}(y;q^{1/2}\,|\,q), \end{aligned} \tag{1.17}$$

see (2.7). We proceed to compute an exact formula for $\Psi_1(x,y\,|\,q)$ in Section 3. In Section 3 we also show that $\Psi_n(x,y\,|\,q)$ is a q-convolution kernel where the translation used is the q-translation introduced in [15]. Section 4 is devoted to proving the important formula

$$\mathscr{D}_{q,x}^{-1}\Psi_\alpha(x,y\,|\,q) = \Psi_{\alpha+1}(x,y\,|\,q), \tag{1.18}$$

where $\mathscr{D}_{q,x}^{-1}$ is the inverse to the Askey–Wilson operator, defined in (4.4). The use of formula (1.18) and the expression for $\Psi_1(x,y\,|\,q)$ then allows us to find the subsequent q-polynomials $\Psi_2(x,y\,|\,q), \Psi_3(x,y\,|\,q), \ldots$. In Section 5 we introduce our version of the q-Bernoulli polynomials with particular attention to the odd and even ones and show that they contain the one variable q-Bernoulli polynomials defined by Suslov [22]. In Section 6 we derive an expression for the kernel $K(x,y;\mathbf{a})$ of the inverse Askey–Wilson operator $\mathscr{D}_q^{-1}$ containing four parameters a, b, c, d and show that (4.4) follows from it. In Section 7 we compute $\mathscr{D}_q^{-1}$ on some q-polynomials which are analogues of $(x-y)^k$. This computation is then used to establish parity properties of the q-Bernoulli polynomials introduced in Section 5. We close the paper with an Appendix where we evaluate a particular kind of Askey–Wilson type integral and derive a very useful theta function identity. The evaluated integral is displayed in (8.4)–(8.5) while the theta function identity is (8.7).

It must be noted that Al-Salam [1] introduced a different q-fractional integral operator which corresponds to q-calculus based on the operator D_q,

$$(D_q)\, f(x) := \frac{f(x) - f(qx)}{x - qx}, \qquad x \neq 0, \quad (D_q)\, f(0) = f'(0). \tag{1.19}$$

This line of investigation was continued by Al-Salam and Verma where a q-Leibniz rule was established in [2]. See also [3].

2. PRELIMINARIES

In [13] Ismail introduced the system of polynomials $\{h_{n,\nu}(x; q)\}$ by

$$h_{-1,\nu}(x; q) = 0, \qquad h_{0,\nu}(x; q) = 1, \tag{2.1}$$

$$2x(1 - q^{\nu+n})\, h_{n,\nu}(x; q) = h_{n+1,\nu}(x; q) + q^{n+\nu-1} h_{n-1,\nu}(x; q). \tag{2.2}$$

These are q-analogues of the Lommel polynomials [25]. He also proved that $J^{(2)}_{\nu+m}(z; q)$ is expressed in terms of $J^{(2)}_{\nu}(z; q)$ and $J^{(2)}_{\nu-1}(z; q)$ as

$$\begin{aligned} &q^{m\nu+m(m-1)/2} J^{(2)}_{\nu+m}(z; q) \\ &\quad = h_{m,\nu}(1/z; q)\, J^{(2)}_{\nu}(z; q) - h_{m-1,\nu+1}(1/z; q)\, J^{(2)}_{\nu-1}(z; q), \qquad m = 1, 2, \ldots. \end{aligned} \tag{2.3}$$

Moreover he proved that the zeros of $z^{-\nu} J^{(2)}_{\nu}(z; q)$, which are symmetric about $z = 0$, are real and simple for $\nu > -1$. Denote by $\{j_{\nu,k}(q)\}$ the sequence of positive zeros of $J^{(2)}_{\nu}(z; q)$, and let $j_{\nu,-k}(q)$ be $-j_{\nu,k}(q)$, for all $k > 0$. Following [13] we define $A_n(\nu+1)$ via

$$\begin{aligned} \frac{d}{dz} J^{(2)}_{\nu}(z; q)\big|_{z = j_{\nu,n}(q)} &= -2 \frac{J^{(2)}_{\nu+1}(j_{\nu,n}(q); q)}{A_n(\nu+1)} \\ &= 2q^{-\nu} \frac{J^{(2)}_{\nu-1}(j_{\nu,n}(q); q)}{A_n(\nu+1)}. \end{aligned} \tag{2.4}$$

The second equality follows from the three term recurrence relation [13]

$$q^{\nu} J^{(2)}_{\nu+1}(z; q) = \frac{2(1 - q^{\nu})}{z} J^{(2)}_{\nu}(z; q) - J^{(2)}_{\nu-1}(z; q). \tag{2.5}$$

With these notations the polynomials $\{h_{n,\nu}(x;q)\}$ satisfy the orthogonality relation (1.16), [13]. Moreover

$$\sum_{k=1}^{\infty} \frac{zA_k(\nu+1)}{z^2-j_{\nu,k}^2(q)} = -2\,\frac{J_{\nu+1}^{(2)}(z;q)}{J_\nu^{(2)}(z;q)}. \tag{2.6}$$

Bustoz and Suslov [9] used the notation

$$\omega_0 = 0, \qquad \omega_n = \tfrac{1}{2}\, j_{1/2,n}(q), \quad n \neq 0, \tag{2.7}$$

so that $\omega_{-n} = -\omega_n$. Using (1.8) and a very lengthy argument they proved

$$\int_{-1}^{1} \mathscr{E}_q(x;i\omega_m)\,\overline{\mathscr{E}_q(x;i\omega_n)}\,w(x;q^{1/2}\,|\,q)\,dx = \frac{\delta_{m,n}}{\pi_n}, \tag{2.8}$$

with

$$\pi_{\pm n} = \sqrt{q}\,\frac{A_n(3/2)}{8\pi\omega_{|n|}}\left[\frac{(-q\omega_n^2;q^2)_\infty}{J_{-1/2}^{(2)}(2\omega_{|n|};q)}\right]^2, \qquad n>0, \tag{2.9}$$

$$\pi_0 = \frac{1}{2\pi}\,\frac{(q,q;q)_\infty}{(q^{1/2},q^{3/2};q)_\infty}.$$

Ismail [14] gave a simple proof of (2.8)–(2.9) and realized the connection between the $\mathscr{E}_q$ orthogonality and the dual orthogonality of the polynomials $\{h_{n,\nu}(x;q)\}$.

Recall that the q-gamma function [4]

$$\Gamma_q(z) = (1-q)^{1-z}\,\frac{(q;q)_\infty}{(q^z;q)_\infty} \tag{2.10}$$

satisfies $\Gamma_q(z) \to \Gamma(z)$ when $q \to 1^-$. Moreover [13]

$$\lim_{q\to 1^-} \frac{j_{\nu,n}(q)}{1-q} = j_{\nu,n}, \qquad \lim_{q\to 1^-} \frac{A_n(\nu)}{1-q} = 2, \quad \nu > -1. \tag{2.11}$$

Therefore $\pi_n \to 1/4$ as $q \to 1^-$, for all $n = 0, \pm 1, \ldots$.

3. Q-FRACTIONAL INTEGRALS

Let

$$f(x) \sim \sum_{-\infty}^{\infty} f_n \mathscr{E}_q(x;i\omega_n). \tag{3.1}$$

In analogy with (1.1) and taking into account (2.8) we define

$$I_{\alpha,q} f(x) = \frac{(1-q)^\alpha}{2^\alpha q^{\alpha/4}} \sum_{n\neq 0} \frac{f_n}{(i\omega_n)^\alpha} \mathscr{E}_q(x; i\omega_n), \qquad \alpha > 0. \tag{3.2}$$

The right-hand side of (3.2) is in $L^2[w(.; q^{1/2} \mid q)]$ since (3.1) implies $\{f_n/\sqrt{\pi_n}\} \in l^2$ and $\omega_n \to \infty$ as $n \to \infty$. The orthogonality relations (2.7)–(2.9) imply

$$I_{\alpha,q} f(x) = \pi_0 \int_{-1}^{1} \Psi_\alpha(x, y \mid q)\, f(y)\, w(x; q^{1/2} \mid q)\, dy, \tag{3.3}$$

where

$$\Psi_\alpha(x, y \mid q) = \frac{(1-q)^\alpha}{2^\alpha q^{\alpha/4} \pi_0} \sum_{n\neq 0} \frac{\pi_n}{(i\omega_n)^\alpha} \mathscr{E}_q(x; i\omega_n)\, \overline{\mathscr{E}_q(y; i\omega_n)}, \tag{3.4}$$

with π_0 and π_n as in (2.9) and $i = e^{i\pi/2}$. The reason for introducing the factor π_0 is to be consistent with (1.2).

The semigroup property

$$I_{\alpha,q} I_{\beta,q} = I_{\alpha+\beta,q}, \qquad \alpha > 0, \quad \beta > 0. \tag{3.5}$$

follows from the definition (3.2) when $\alpha > 0$ and $\beta > 0$. Furthermore (1.8) and (3.2) give

$$\mathscr{D}_q I_{\alpha,q} = I_{\alpha-1,q}, \qquad \alpha > 1, \tag{3.6}$$

which is equivalent to

$$\mathscr{D}_q \Psi_\alpha(x, y \mid q) = \Psi_{\alpha-1}(x, y \mid q), \qquad \alpha > 1, \tag{3.7}$$

Therefore $\mathscr{D}_q^n I_{\alpha,q} = I_{\alpha-n,q}$ if $\alpha > n$.

Suslov [23] proved that

$$\begin{aligned} &\mathscr{E}_q(\cos\theta; \alpha)\, \mathscr{E}_q(\cos\phi; \beta) \\ &\quad = \frac{(\beta^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}\beta^n}{(q; q)_n} e^{-in\phi} \\ &\quad \times (-q^{(1-n)/2} e^{i(\theta+\phi)}\alpha/\beta, -q^{(1-n)/2} e^{i(\phi-\theta)}\alpha/\beta; q)_n \\ &\quad \times {}_2\phi_2\left(\begin{matrix} q^{-n}, \alpha^2/\beta^2 \\ -q^{(1-n)/2} t e^{i(\theta+\phi)}\alpha/\beta, -q^{(1-n)/2} t e^{i(\phi-\theta)} \end{matrix} \,\middle|\, q, e^{2i\phi} \right). \end{aligned} \tag{3.8}$$

The case $\alpha = -\beta$ in (3.8) gives

$$\mathscr{E}_q(x; \alpha)\, \mathscr{E}_q(y; -\alpha) = \mathscr{E}_q(x, -y; \alpha), \tag{3.9}$$

hence (3.4) has the alternate form

$$\Psi_\alpha(x, y \mid q) = \frac{(1-q)^\alpha}{2^\alpha q^{\alpha/4} \pi_0} \sum_{n \neq 0} \frac{\pi_n}{(i\omega_n)^\alpha} \mathscr{E}_q(x, -y; i\omega_n), \tag{3.10}$$

which exihibits the symmetry of $\Psi_\alpha(x, y \mid q)$

$$\Psi_\alpha(x, y \mid q) = \Psi_\alpha(-y, -x \mid q).$$

Furthermore $\mathscr{E}_q(0; \alpha) = 1$ and (3.9) imply the symmetry

$$\Psi_\alpha(y, x \mid q) = e^{-i\pi\alpha} \Psi_\alpha(x, y \mid q). \tag{3.11}$$

It must be emphasized that the series (1.7) defining $\mathscr{E}_q(x, y; \alpha)$ converges for $x, y \in [-1, 1]$ and $|\alpha| < 1$. The $\mathscr{E}_q$ in (3.10) is the analytic continuation in α of (1.7) discussed in [18].

We now establish the alternate representation of $\Psi_\alpha(x, y \mid q)$ given in (1.17). Using (1.14) we find

$$\begin{aligned}
&\frac{2^\alpha q^{\alpha/4}}{(1-q)^\alpha} \Psi_\alpha(x, y \mid q) \\
&\quad = \frac{\sqrt{q}}{4(1-(\sqrt{q}))} \sum_{n \neq 0} \frac{A_{|n|}(3/2)/\omega_n^2}{(i\omega_n)^\alpha \left[J^{(2)}_{-1/2}(2\omega_n; q)\right]^2} \\
&\qquad \times \sum_{r, s=0}^{\infty} i^{r-s} q^{(r^2+s^2)/4} (1-q^{r+1/2})(1-q^{s+1/2})\, J^{(2)}_{r+1/2}(2\omega_n; q) \\
&\qquad \times J^{(2)}_{s+1/2}(2\omega_n; q)\, C_r(x; q^{1/2} \mid q)\, C_s(y; q^{1/2} \mid q) \\
&\quad = \frac{\sqrt{q}\, e^{-i\pi\alpha/2}}{4(1-\sqrt{q})} \sum_{r, s=1}^{\infty} \sum_{n \neq 0} \frac{A_{|n|}(3/2)\, i^{r-s}}{\omega_n^{2+\alpha}} \frac{(1-q^{r+1/2})(1-q^{s+1/2})}{q^{(r^2+s^2)/4}} \\
&\qquad \times h_{r-1, 3/2}(1/(2\omega_n); q)\, h_{s-1, 3/2}(1/(2\omega_n); q) \\
&\qquad \times C_r(x; q^{1/2} \mid q)\, C_s(y; q^{1/2} \mid q).
\end{aligned}$$

This proves (1.17).

We now study the kernel $\Psi_\alpha(x, y\,|\,q)$ when $\alpha = 1, 2, \ldots$. Clearly for $\alpha = 1$ we can use the relationships (2.1)–(2.2) and the orthogonality relation (2.6) to justify

$$\frac{2q^{1/4}}{(1-q)}\Psi_1(x, y\,|\,q) = \frac{-i}{(1-\sqrt{q}\,)}\sum_{r,s=0}^{\infty} C_{r+1}(x; q^{1/2}\,|\,q)\, C_{s+1}(y; q^{1/2}\,|\,q) \times \frac{i^{r-s}(1-q^{s+3/2})}{q^{(r^2+s^2+2r+2s)/4}} \sum_{n\neq 0} \frac{A_{|n|}(3/2)}{(2\omega_n)^2} h_s(1/(2\omega_n); q) \times [h_{r+1,3/2}(1/(2\omega_n); q) + q^{r+1/2} h_{r-1,3/2}(1/(2\omega_n); q)].$$

This and the orthogonality relation (2.6) lead to the representation

$$\frac{2}{(1+\sqrt{q}\,)}\Psi_1(x, y\,|\,q) = \sum_{r=1}^{\infty} q^{r/2}[C_{r+1}(x; q^{1/2}\,|\,q)\, C_r(y; q^{1/2}\,|\,q) - C_{r+1}(y; q^{1/2}\,|\,q)\, C_r(x; q^{1/2}\,|\,q)]. \tag{3.12}$$

We now simplify (3.12). Let

$$G(x, y) := \sum_{r=0}^{\infty} q^{r/2}[C_{r+1}(x; q^{1/2}\,|\,q)\, C_r(y; q^{1/2}\,|\,q) - C_{r+1}(y; q^{1/2}\,|\,q)\, C_r(x; q^{1/2}\,|\,q)].$$

The orthogonality relation (1.12), the Christoffel–Darboux formula [24] and the observation that the coefficient of x^n in $C_n(x; \beta\,|\,q)$ is $2^n(\beta; q)_n/(q; q)_n$ implies

$$\frac{G(\cos\theta, \cos\phi)}{2(x-y)} = \sum_{n=0}^{\infty} \frac{1-q^{1/2}}{1-q^{n+1}} q^{n/2} \sum_{k=0}^{n} \frac{1-q^{k+1/2}}{1-q^{1/2}} C_k(\cos\theta; q^{1/2}\,|\,q)\, C_k(\cos\theta; q^{1/2}\,|\,q)$$
$$= \sum_{j=0}^{\infty} \frac{1-q^{1/2}}{1-q^{j+1/2}} q^j \times \sum_{k=0}^{\infty} \frac{1-q^{k+1/2}}{1-q^{1/2}} q^{k(j+1/2)} C_k(\cos\theta; q^{1/2}\,|\,q)\, C_k(\cos\theta; q^{1/2}\,|\,q).$$

Exercise 7.33(ii) in [12] evaluates the k-sum above as

$$\frac{(q^{\frac{1}{2}}, q^{2j+1}; q)_\infty}{(q, q^{2j+\frac{5}{2}}; q)_\infty} \frac{(q^{j+1}e^{i(\theta+\phi)}, q^{j+1}e^{-i(\theta+\phi)}, q^{j+2}e^{i(\theta-\phi)}, q^{j+2}e^{i(\phi-\theta)}; q)_\infty}{(q^{j+\frac{1}{2}}e^{i(\theta+\phi)}, q^{j+\frac{1}{2}}e^{-i(\theta+\phi)}, q^{j+\frac{1}{2}}e^{i(\theta-\phi)}, q^{j+\frac{1}{2}}e^{i(\phi-\theta)}; q)_\infty}$$
$$\times {}_8W_7(q^{2j+\frac{3}{2}}; q^{\frac{1}{2}}, q^{j+\frac{3}{2}}e^{i(\theta+\phi)}, q^{j+\frac{3}{2}}e^{-i(\theta+\phi)}, q^{j+\frac{1}{2}}e^{i(\theta-\phi)}, q^{j+\frac{1}{2}}e^{i(\phi-\theta)} \mid q, q^{\frac{1}{2}}).$$

Therefore we find

$$\frac{G(\cos\theta, \cos\phi)}{2(x-y)}$$
$$= \frac{(q^{\frac{1}{2}}, q^2; q)_\infty}{(q, q^{\frac{5}{2}}; q)_\infty} \frac{(qe^{i(\theta+\phi)}, qe^{-i(\theta+\phi)}, q^2e^{i(\theta-\phi)}, q^2e^{i(\phi-\theta)}; q)_\infty}{(q^{\frac{1}{2}}e^{i(\theta+\phi)}, q^{\frac{1}{2}}e^{-i(\theta+\phi)}, q^{\frac{1}{2}}e^{i(\theta-\phi)}, q^{\frac{1}{2}}e^{i(\phi-\theta)}; q)_\infty}$$
$$\times \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{1-q^{\frac{1}{2}}}{1-q^{j+\frac{1}{2}}} q^j(1-q^{2j+1}) \frac{(1-q^{\frac{1}{2}}e^{i(\theta+\phi)})(1-q^{\frac{1}{2}}e^{-i(\theta+\phi)})}{(1-q^{j+\frac{1}{2}}e^{i(\theta+\phi)})(1-q^{j+\frac{1}{2}}e^{-i(\theta+\phi)})}$$
$$\times \frac{1-q^{2j+2k+3/2}}{1-q^{3/2}} q^{k/2} \frac{(q^{3/2}; q)_{2j+k}\,(q^{1/2}; q)_k}{(q^2; q)_{2j+k}\,(q; q)_k}$$
$$\times \frac{(q^{3/2}e^{i(\theta+\phi)}, q^{3/2}e^{-i(\theta+\phi)}, q^{1/2}e^{i(\theta-\phi)}, q^{1/2}e^{i(\phi-\theta)}; q)_{j+k}}{(qe^{i(\theta+\phi)}, qe^{-i(\theta+\phi)}, q^2e^{i(\theta-\phi)}, q^2e^{i(\phi-\theta)}; q)_{j+k}}$$
$$= (q^{1/2}; q)_2 \frac{(qe^{i(\theta+\phi)}, qe^{-i(\theta+\phi)}, q^2e^{i(\theta-\phi)}, q^2e^{i(\phi-\theta)}; q)_\infty}{(q^{\frac{1}{2}}e^{i(\theta+\phi)}, q^{\frac{1}{2}}e^{-i(\theta+\phi)}, q^{\frac{1}{2}}e^{i(\theta-\phi)}, q^{\frac{1}{2}}e^{i(\phi-\theta)}; q)_\infty}$$
$$\times \sum_{n=0}^\infty \frac{1-q^{2n+3/2}}{1-q^{3/2}} \frac{(q^{3/2}, q^{1/2}; q)_n}{(q, q^2; q)_n}$$
$$\times \frac{(q^{3/2}e^{i(\theta+\phi)}, q^{3/2}e^{-i(\theta+\phi)}, q^{1/2}e^{i(\theta-\phi)}, q^{1/2}e^{i(\phi-\theta)}; q)_n}{(qe^{i(\theta+\phi)}, qe^{-i(\theta+\phi)}, q^2e^{i(\theta-\phi)}, q^2e^{i(\phi-\theta)}; q)_n} q^{n/2}$$
$$\times {}_8\phi_7\left(\begin{matrix} q, q^{\frac{3}{2}}, -q^{\frac{3}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}e^{i(\theta+\phi)}, q^{\frac{1}{2}}e^{-i(\theta+\phi)}, q^{n+\frac{3}{2}}, q^{-n} \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^{\frac{3}{2}}e^{-i(\theta+\phi)}, q^{\frac{3}{2}}e^{i(\theta+\phi)}, q^{-n+\frac{1}{2}}, q^{n+2} \end{matrix} \middle| q, q\right).$$

The ${}_8\phi_7$ series is summable by Jackson's formula [12, (II.22)] and its sum is given by

$$\frac{(q^2, qe^{i(\theta+\phi)}, qe^{-i(\theta+\phi)}, q; q)_n}{(q^{3/2}, q^{3/2}e^{i(\theta+\phi)}, q^{3/2}e^{-i(\theta+\phi)}, q^{1/2}; q)_n}.$$

Hence the n-sum is

$$ {}_6\phi_5\left(\begin{matrix} q^{3/2}, q^{7/4}, -q^{7/4}, q, q^{1/2}e^{i(\phi-\theta)}q^{1/2}e^{i(\theta-\phi)} \\ q^{3/4}, -q^{3/4}, q^{3/2}, q^2e^{i(\theta-\phi)}, q^2e^{i(\phi-\theta)} \end{matrix} \middle| q, q^{1/2}\right), \tag{3.13}$$

whose sum is

$$\frac{(qe^{i(\theta-\phi)}, qe^{i(\phi-\theta)}, q^{3/2}, q^{5/2}; q)_\infty}{(q^{3/2}, q^{1/2}, q^2e^{i(\theta-\phi)}, q^2e^{i(\phi-\theta)}; q)_\infty},$$

by [12, (II.20)]. This establishes the explicit form

$$\Psi_1(x, y \mid q) = \tfrac{1}{4}(y-x) - A_1(e^{i\theta}, e^{i\phi}), \tag{3.14}$$

where

$$A_1(e^{i\theta}, e^{i\phi}) = -\frac{(1+q^{1/2})(\cos\theta-\cos\phi)(qe^{i(\theta+\phi)}, qe^{-i(\theta+\phi)}, qe^{i(\theta-\phi)}, qe^{i(\phi-\theta)}; q)_\infty}{4(q^{1/2}e^{i(\theta+\phi)}, q^{1/2}e^{-i(\theta+\phi)}, q^{1/2}e^{i(\theta-\phi)}, q^{1/2}e^{i(\phi-\theta)}; q)_\infty}, \tag{3.15}$$

and $x = \cos\theta$, $y = \cos\phi$. It is easy to see that A_1 is annihilated by the Askey–Wilson operator $\mathcal{D}_q$. Thus Ψ_1 is a q-polynomial in $x-y$ of degree one. In the next section we study $\Psi_m(x, y \mid q)$ for $m = 2, 3, \ldots$.

The limit as $q \to 1^-$ of A_1 can be found from the q-binomial theorem, since

$$\lim_{q\to 1^-} \frac{(qe^{i(\pm\theta\pm\phi)}; q)_\infty}{(q^{1/2}e^{i(\pm\theta\pm\phi)}; q)_\infty} = \lim_{q\to 1^-} {}_1\phi_0(q^{1/2}; -; q, q^{1/2}e^{i(\pm\theta\pm\phi)}) = [1-e^{i(\pm\theta\pm\phi)}]^{-1/2}. \tag{3.16}$$

Therefore

$$\lim_{q\to 1^-} A_1(e^{i\theta}, e^{i\phi}) = \frac{-2(\cos\theta-\cos\phi)}{4[\{1-2xe^{i\phi}+e^{2i\phi}\}\{1-2xe^{-i\phi}+e^{-2i\phi}\}]^{1/2}} = \frac{\cos\phi-\cos\theta}{4\,|\cos\theta-\cos\phi|},$$

that is,

$$\lim_{q\to 1^-} A_1(e^{i\theta}, e^{i\phi}) = \tfrac{1}{4}\,\mathrm{sgn}(y-x). \tag{3.17}$$

Suslov's results in [22] correspond to the special case $y = y_0$, with $y_0 := (q^{1/4} + q^{-1/4})/2$. To see this consider $\mathscr{E}_q(y_0; i\omega_m)$, for $m \neq 0$. The relationships (1.7) and (1.9) give

$$\begin{aligned}
\mathscr{E}_q(y_0; i\omega_m) &= \frac{(-\omega_m^2; q^2)_\infty}{(-q\,\omega_m^2; q^2)_\infty} \sum_{n=0}^{\infty} (-iq^{(-n+3/2)/2}, -iq^{(-n+1/2)/2}; q)_n \frac{\omega_m^n q^{n^2/4}}{(q; q)_n} \\
&= \frac{(-\omega_m^2; q^2)_\infty}{(-q\,\omega_m^2; q^2)_\infty} \sum_{n=0}^{\infty} (-iq^{(-n+1/2)/2}; q^{1/2})_{2n} \frac{\omega_m^n q^{n^2/4}}{(q; q)_n} \\
&= \frac{(-\omega_m^2; q^2)_\infty}{(-q\,\omega_m^2; q^2)_\infty} \sum_{n=0}^{\infty} (-iq^{1/4}, iq^{1/4}; q^{1/2})_n \frac{\omega_m^n i^n}{(q; q)_n} \\
&= \frac{(-\omega_m^2; q^2)_\infty}{(-q\,\omega_m^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q^{1/2}; q)_n}{(q; q)_n} i^n \omega_m^n \\
&= \frac{(-\omega_m^2; q^2)_\infty}{(-q\,\omega_m^2; q^2)_\infty} \frac{(-i\omega_m q^{1/2}; q)_\infty}{(i\omega_m; q)_\infty}.
\end{aligned}$$

Therefore $\mathscr{E}_q(y_0; i\omega_m)$ is given by

$$\mathscr{E}_q(y_0; i\omega_m) = \frac{(-i\omega_m; q)_\infty}{(iq^{1/2}\omega_m; q)_\infty}. \tag{3.18}$$

Finally $J^{(2)}_{1/2}(2\omega_m; q) = 0$ implies $(i\omega_m; \sqrt{q})_\infty = (-i\omega_m; \sqrt{q})_\infty$, hence

$$\mathscr{E}_q(y_0; i\omega_m) = \sqrt{\frac{(-\omega_m^2; q^2)_\infty}{(-q\omega_m^2; q^2)_\infty}}, \tag{3.19}$$

and $\Psi_\alpha(x, y_0 \mid q)$ becomes

$$\begin{aligned}
\Psi_\alpha(x, y_0 \mid q) = {} & \frac{2\pi(1-q)^\alpha (\sqrt{q}, \sqrt{q}; q)_\infty}{2^\alpha q^{\alpha/4}(q, q; q)_\infty} \\
& \times \sum_{n \neq 0} \frac{\pi_n}{(i\omega_n)^\alpha} \sqrt{\frac{(-\omega_m^2; q^2)_\infty}{(-q\omega_m^2; q^2)_\infty}}\, \mathscr{E}_q(x; i\omega_n).
\end{aligned} \tag{3.20}$$

Suslov [22, Sect. 7] used (3.20) to define q-Bernoulli polynomials; see Section 5.

In [15], Ismail introduced a generalized translation operator (q-translation) by defining it first on the continuous q-Hermite polynomials then extending it by linearity to all polynomials. The action of the q-translation by y, E_q^y, on $H_n(x \mid q)$ is defined by

$$E_q^y H_n(x \mid q) := \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q H_m(x \mid q)\, g_{n-m}(y)\, q^{(m^2-n^2)/4}, \tag{3.21}$$

where the polynomials $\{g_n(x)\}$ are

$$\frac{g_n(x)}{(q;q)_n} = \sum_{k=0}^{[n/2]} \frac{q^k}{(q^2;q^2)_k} \frac{H_{n-2k}(x \mid q)}{(q;q)_{n-2k}}\, q^{(n-2k)^2/4}. \tag{3.22}$$

It was shown that E_q^y commutes with $\mathscr{D}_q$, and E_q^y satisfies

$$E_q^y \mathscr{E}_q(x;\alpha) = \mathscr{E}_q(x;\alpha)\, \mathscr{E}_q(y;\alpha) = \mathscr{E}_q(x, y;\alpha). \tag{3.23}$$

Thus

$$\Psi_\alpha(x, y \mid q) = E_q^{-y} \Psi_\alpha(x, 0 \mid q) = E_q^x \Psi(0, y \mid q). \tag{3.24}$$

This means that $\Psi_\alpha(x, y \mid q)$ is a q-analogue of the convolution kernel $\Psi_\alpha(x-y)$ of (1.2).

Observe that (3.23) implies $\lim_{q\to 1} g_n(x) = (2x)^n$, since $\lim_{q\to 1} H_n(x \mid q) = (2x)^n$. Also note that E_q^y has the operational representation

$$E_q^y = \mathscr{E}_q(y; ((1-q)\, q^{-1/4}/2)\, \mathscr{D}_q). \tag{3.25}$$

In particular the q-constants are invariant under q-translations.

4. Q-INVERSE OPERATORS

The main result of this section is

$$\mathscr{D}_q^{-1} \Psi_\alpha(x, y \mid q) = \Psi_{\alpha+1}(x, y \mid q), \qquad \alpha > 0, \tag{4.1}$$

where $\mathscr{D}_q^{-1}$ is the integral operator defined below in (4.4).

Applying (2.1) and (2.2) we see that

$$\frac{1-q^{r+3/2}}{\omega_n} h_{r,3/2}(1/(2\omega_n); q) = h_{r+1,3/2}(1/(2\omega_n); q) + q^{r+1/2} h_{r-1,3/2}(1/(2\omega_n); q),$$

which we substitute in (3.12), then replace r in the r sum that contains $h_{r-1,3/2}$ by $r+1$. After combining the two r sums we get to the equivalent representation

$$\begin{aligned}
&\Psi_{\alpha+1}(x, y \mid q)\\
&\quad= \frac{2^{-\alpha-3}(1-q)^{\alpha+1}}{(1-\sqrt{q})\, q^{\alpha/4}}\, e^{-i\pi\alpha/2} \sum_{s=0}^{\infty} \sum_{n\neq 0} \frac{A_{|n|}(3/2)\, i^{-s}(1-s^{s+3/2})}{\omega_n^{\alpha+2} q^{s(s+2)/4}}\\
&\quad\times h_{s,3/2}(1/(2\omega_n); q)\, C_{s+1}(y; q^{1/2} \mid q)\\
&\quad+ \frac{2^{-\alpha-3}(1-q)^{\alpha+1}}{(1-\sqrt{q})\, q^{\alpha/4}}\, e^{-i\pi\alpha/2} \sum_{r,s=0}^{\infty} \sum_{n\neq 0} \frac{A_{|n|}(3/2)\, i^{r-s}(1-s^{s+3/2})}{\omega_n^{\alpha+2} q^{(r^2+s(s+2))/4}}\\
&\quad\times h_{r,3/2}(1/(2\omega_n); q)\, h_{s,3/2}(1/(2\omega_n); q)\\
&\quad\times [\sqrt{q}\, C_{r+2}(x; q^{1/2} \mid q) - C_r(x; q^{1/2} \mid q)]. \qquad (4.2)
\end{aligned}$$

THEOREM 4.1. *The right inverse to the Askey–Wilson operator satisfies*

$$\mathscr{D}_{q,x}^{-1} \Psi_\alpha(x, y \mid q) = \Psi_{\alpha+1}(x, y \mid q), \qquad (4.3)$$

where

$$\begin{aligned}
(\mathscr{D}_q^{-1} f)(\cos\theta) &= \frac{(1-q)(q;q)_\infty^2}{4\pi q^{1/4}}\, h(\cos\theta; -q^{1/4}, -q^{3/4})\\
&\quad\times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty\, f(\cos\psi)\, d\psi}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})}\\
&\quad- \frac{(q;q)_\infty^2}{4\pi q^{1/4}(q^{1/2}, q^{3/2}; q)_\infty}\\
&\quad\times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty\, [1-2q^{1/4}\cos\psi + q^{1/2}]}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{1/4}, q^{3/4})}\, f(\cos\psi)\, d\psi. \qquad (4.4)
\end{aligned}$$

The origin of this particular parameter free form of the kernel for $\mathscr{D}_q^{-1}$ will be made clear in Section 6. Furthermore the second term on the right-hand side is a constant and corresponds to specifying the lower limit of integration in a definite integral.

We next prove Theorem 4.1.

Proof. We first evaluate $\mathscr{D}_q^{-1} C_n(x; q^{1/2} \mid q)$. We have

$$\int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty\, C_{r+1}(\cos\psi; q^{1/2} \mid q)\, d\psi}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})}$$

$$= \int_0^\pi \frac{q^{-(r+1)/4}(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})}$$

$$\times {}_4\phi_3\left(\begin{matrix} q^{-r-1}, q^{r+2}, q^{1/4}e^{i\psi}, q^{1/4}e^{-i\psi} \\ -q^{1/2}, \quad q, \quad -q \end{matrix} \,\middle|\, q, q\right) d\psi$$

$$= \int_0^\pi \frac{(-1)^{r+1}\, q^{-(r+1)/4}(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})}$$

$$\times {}_4\phi_3\left(\begin{matrix} q^{-r-1}, q^{r+2}, -q^{1/4}e^{i\psi}, -q^{1/4}e^{-i\psi} \\ -q^{1/2}, \quad q, \quad -q \end{matrix} \,\middle|\, q, q\right) d\psi$$

$$= \sum_{k=0}^{r+1} \frac{(q^{-r-1}, q^{r+2}; q)_k\, q^k}{(q, q, -q, -q^{1/2}; q)_k} \frac{2\pi(q^{k+2}; q)_\infty\, (-1)^{r+1}\, q^{-(r+1)/4}}{(q, q, q^{k+1}; q)_\infty\, h(\cos\theta; -q^{k+3/4}, -q^{5/4})}$$

$$= \frac{2\pi(-1)^{r+1}\, q^{-(r+1)/4}}{(1-q)(q, q; q)_\infty\, h(\cos\theta; -q^{3/4}, -q^{5/4})}$$

$$\times {}_4\phi_3\left(\begin{matrix} q^{-r-1}, q^{r+2}, -q^{3/4}e^{i\theta}, -q^{3/4}e^{-i\theta} \\ -q^{1/2}, \quad q^2, \quad -q \end{matrix} \,\middle|\, q, q\right),$$

where the Sears transformation [12, (III.15)] was used in going from the second line to the third step. Furthermore the Askey–Wilson integral evaluation was used to go from the integral involving a ${}_4\phi_3$ to the k sum. So the first term on the right-hand side of (4.4) with $f(x) = C_{r+1}(x; q^{1/2} \mid q)$ is

$$\frac{1 + 2xq^{1/4} + q^{1/2}}{2(-1)^{r+1}\, q^{(r+2)/4}}\, {}_4\phi_3\left(\begin{matrix} q^{-r-1}, q^{r+2}, -q^{3/4}e^{i\theta}, -q^{3/4}e^{-i\theta} \\ -q^{1/2}, \quad q^2, \quad -q \end{matrix} \,\middle|\, q, q\right). \tag{4.5}$$

To the ${}_4\phi_3$ in (4.5) we apply the Sears transformation [12, (III.15)] with $n = r+1$, $a = -q^{3/4}e^{-i\theta}$, $d = q^2$, then transform the answer again by the same transformation with $n = r$, $a = -q^{3/4}e^{-i\theta}$, $d = q^2$. Finally apply another Sears transformation with $n = r$, $a = q^{r+3}$, $d = -q^{3/2}$. This identifies the ${}_4\phi_3$ in (4.5) as

$$(-1)^r\, q^{1/2} \frac{1 - 2\cos\theta q^{1/4} + q^{1/2}}{(1+q)(1+q^{1/2})}\, {}_4\phi_3\left(\begin{matrix} q^{-r}, q^{r+3}, q^{3/4}e^{i\theta}, q^{3/4}e^{-i\theta} \\ q^2, \quad -q^2, \quad -q^{3/2} \end{matrix} \,\middle|\, q, q\right),$$

which is

$$(-1)^r \frac{1-2q^{1/4}\cos\theta+q^{1/2}}{(1+q)(1+q^{1/2})} \frac{(q;q)_r}{(q^3;q)_r} q^{(3r+2)/4} C_r(\cos\theta; q^{3/2} \mid q). \tag{4.6}$$

Hence the first term on the right-hand side of (4.4) with $f = C_{r+1}$ is

$$\frac{q^{r/2}(q;q)_r}{2(q^3;q)_r} \frac{(1-2q^{1/4}\cos\theta+q^{1/2})(1+2q^{1/4}\cos\theta+q^{1/2})}{(1+q)(1+q^{1/2})} \times C_r(\cos\theta; q^{3/2} \mid q)$$

which reduces to

$$\frac{(1-q)\,q^{r/2}}{2(1-q^{r+3/2})} [q^{1/2}C_{r+2}(\cos\theta; q^{1/2} \mid q) - C_r(\cos\theta; q^{1/2} \mid q)], \tag{4.7}$$

where we used (3.8) of [18] in the last step. Therefore the contribution to $\mathscr{D}_q^{-1}\Psi_\alpha(\cos\theta, y \mid q)$ from the first term in (4.3) is

$$\frac{(1-q)^{\alpha+1}q^{-\alpha/4}}{2^{\alpha+3}(1-q^{1/2})} e^{-i\pi\alpha/2} \sum_{r,s=0}^{\infty} \frac{A_{|n|}(3/2)(1-q^{s+3/2})\, i^{r-s}}{\omega_n^{\alpha+2} q^{(r^2+s(s+2))/4}} h_{s,3/2}(1/(2\omega_n); q) \times h_{r,3/2}(1/(2\omega_n); q)\, C_{s+1}(y; q^{1/2} \mid q) \times [q^{1/2}C_{r+2}(\cos\theta; q^{1/2} \mid q) - C_r(\cos\theta; q^{1/2} \mid q)],$$

which is precisely the second term on the right-hand side of (4.2). Since $C_1(x; q^{1/2} \mid q) = 2x/(1+q^{1/2})$, we find that the second term in (4.4) is

$$\frac{(1-q)(q,q;q)_\infty}{4\pi(q^{1/2},q^{1/2};q)_\infty} \times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty\, C_1(\cos\psi; q^{1/2} \mid q)\, C_{r+1}(\cos\psi; q^{1/2} \mid q)\, d\psi}{h(\cos\psi; q^{1/4}, q^{3/4}, -q^{1/4}, -q^{3/4})} = \frac{(1-q)}{2(1-q^{1/2})} \delta_{r,0}, \tag{4.8}$$

by the orthogonality relation (1.12). So, from (4.3) and (4.8) it follows that the contribution of the second term on the right-hand side of (4.4) (with $f = C_{r+1}(x; q^{1/2} \mid q)$) to $\mathscr{D}_{q,x}^{-1}\Psi_\alpha(x, y \mid q)$ is

$$\frac{q^{-a/4}(1-q)^{\alpha+1}}{2^{\alpha+3}(1-q^{1/2})} e^{-i\pi\alpha} \sum_{s=0}^{\infty} \sum_{n\neq 0} \frac{A_{|n|}(3/2)\, i^{-s}(1-q^{s+3/2})}{\omega_n^{\alpha+2} q^{s(s+2)/4}} \times C_{s+1}(y; q^{1/2} \mid q)\, h_{s,3/2}(1/(2\omega_n); q),$$

which is of course the first term, on the right-hand side of (4.2). This completes the proof of Theorem 4.1. ▮

We now demonstrate the usefulness of Theorem 4.1 by computing the kernels $\Psi_2(x, y \mid q)$ and $\Psi_3(x, y \mid q)$.

Since

$$y - x = \frac{e^{i\phi} h(\cos\theta; e^{-i\phi}, qe^{i\phi})}{2h(\cos\theta; qe^{-i\phi}, qe^{i\phi})}$$

we find that the first term of the right-hand side of

$$\begin{aligned} \mathscr{D}_{q,x}^{-1}&\Psi_1(x, y \mid q) \\ &= \frac{(1-q)(q, q; q)_\infty}{4\pi q^{1/4}} h(x; -q^{1/4}, -q^{3/4}) \\ &\quad \times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty \Psi_1(\cos\psi, \cos\phi \mid q)\, d\psi}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})} \\ &\quad - \frac{\Gamma_q^2(1/2)\, q^{-1/4}}{4\pi(1+q^{1/2})} \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty \Psi_1(\cos\psi, \cos\phi \mid q)\, d\psi}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{3/4}, q^{5/4})}, \end{aligned} \tag{4.9}$$

can be written as

$$\begin{aligned} &\frac{(1-q)(q, q; q)_\infty}{32\pi q^{1/4}} h(x; -q^{1/4}, -q^{3/4})\, e^{i\phi} \\ &\times \left[\int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty h(\cos\psi; e^{-i\phi}, qe^{i\phi})\, d\psi}{h(\cos\psi; qe^{i\phi}, qe^{-i\phi}, -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})} \right. \\ &\left. - \int_0^\pi \frac{(1+q^{1/2})(e^{2i\psi}, e^{-2i\psi}; q)_\infty h(\cos\psi; e^{-i\phi}, qe^{i\phi})\, d\psi}{h(\cos\psi; q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta}, -q^{1/4}, -q^{3/4})} \right]. \end{aligned} \tag{4.10}$$

By (8.5) the first integral in (4.10) is

$$\frac{2\pi q^{-1/4}e^{-i\phi}}{(1-q^2)(q,q;q)_\infty}\left[\frac{(1+2q^{1/4}y+q^{1/2})(1+2q^{3/4}y+q^{3/2})}{h(x;-q^{1/4},-q^{3/4})}\right.$$
$$\left.-\frac{(1-2q^{1/2}\cos(\theta+\phi)+q)(1-2q^{1/2}\cos(\theta-\phi)+q)}{h(x;-q^{1/4},-q^{3/4})}\right]$$

while the second term is

$$-\frac{2\pi q^{-1/4}e^{-i\phi}(1+q^{1/2})}{(1-q)(q,q;q)_\infty}$$
$$\times\left[\frac{1+2q^{1/4}x+q^{1/2}}{h(x;-q^{1/4},-q^{3/4})}-\frac{(q^{1/2};q)^2_\infty\, h(y;q^{1/4},q^{3/4})}{h(x;qe^{i\phi},qe^{-i\phi})}\right].$$

Thus the first term on the right-hand side of (4.9) is

$$\frac{1+q^{-1/2}}{16}\left[\frac{h(x;-q^{1/4},-q^{3/4})\,h(y;q^{1/4},q^{3/4})}{h(x;qe^{i\phi},qe^{-i\phi})(q^{1/2};q)^{-2}_\infty}-(1+2q^{1/4}x+q^{1/2})\right.$$
$$+\frac{(1+2q^{1/4}y+q^{1/2})(1+2q^{3/4}y+q^{3/2})}{(1+q^{1/2})(1+q)}$$
$$\left.-\frac{(1-2q^{1/2}\cos(\theta+\phi)+q)(1-2q^{1/2}\cos(\theta-\phi)+q)}{(1+q^{1/2})(1+q)}\right]. \tag{4.11}$$

On the other hand the second term on the right-hand side of (4.9) is

$$\frac{\Gamma_q^2(1/2)\,q^{-1/2}}{32\pi(1+q^{1/2})}\int_0^\pi\frac{(e^{2i\psi},e^{-2i\psi};q)_\infty}{h(\cos\psi;q^{3/4},q^{5/4},-q^{1/4},-q^{3/4})}$$
$$\times\{(1+2q^{1/4}\cos\psi+q^{1/2})-(1+2q^{1/4}\cos\phi+q^{1/2})\}\,d\psi$$
$$+\frac{\Gamma_q^2(1/2)\,q^{-1/4}}{32\pi e^{-i\phi}}\int_0^\pi\frac{(e^{2i\psi},e^{-2i\psi};q)_\infty\,h(\cos\psi;e^{-i\phi},qe^{i\phi})\,d\psi}{h(\cos\psi;q^{3/4},q^{5/4},-q^{1/4},-q^{3/4},q^{1/2}e^{i\phi},q^{1/2}e^{-i\phi})}$$
$$=\frac{(1+q^{1/2})(1-q^2)}{16q^{1/2}(1-q^{3/2})}-\frac{(1+q^{-1/2})}{16}(1+2q^{1/4}y+q^{1/2})+\frac{(1+q^{1/2})^2}{16q^{1/2}}$$
$$-\frac{q^{-1/2}}{16(1+q)}(1-2q^{1/4}y+q^{1/2})(1-2q^{3/4}y+q^{3/2}). \tag{4.12}$$

We now use the identity (8.7) on the first term in (4.11), then combine (4.11) and (4.12), simplify, and use Theorem 4.1 to obtain

$$\Psi_2(\cos\theta, \cos\phi \mid q) = \frac{(1+q^{-1/2})}{16}\left\{\frac{h(x; q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi})}{h(x; qe^{i\phi}, qe^{-i\phi})} - \frac{(1-q)\, q^{1/2}}{1-q^{3/2}} - \frac{(1-2q^{1/2}\cos(\theta+\phi)+q)(1-2q^{1/2}\cos(\theta-\phi)+q)}{(1+q)(1+q^{1/2})}\right\}. \tag{4.13}$$

To bring the relationship between $\Psi_1(x, y \mid q)$ and $\Psi_2(x, y \mid q)$ in a sharper focus we may rewrite them in the form

$$\Psi_1(x, y \mid q) = -A_1(e^{i\theta}, e^{i\phi}) - A_0(x-y), \tag{4.14}$$

$$\Psi_2(x, y \mid q) = -A_2 u_0(x, y) - A_1(q^{1/2}e^{i\theta}, e^{i\phi})\, u_1(x, y) - \frac{\sqrt{q}}{1+q} A_0 u_2(x, y), \tag{4.15}$$

where

$$A_0 = 1/4,$$
$$A_1(e^{i\theta}, e^{i\phi}) = -\frac{(1+q^{1/2})}{8e^{-i\theta}} \frac{h(\cos\phi; e^{-i\theta}, qe^{i\theta})}{h(\cos\phi; q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})}, \tag{4.16}$$
$$A_2 = \frac{(1+q^{1/2})(1-q)}{16(1-q^{3/2})},$$

and the u-functions are defined in (5.4)–(5.8). In a fairly analogous manner we may now compute $\Psi_3(x, y \mid q)$ by applying $\mathscr{D}_q^{-1}$ on (4.15). Thus we get

$$\Psi_3(x, y \mid q) = -A_2 u_1(x, y) - \frac{\sqrt{q}}{1+q} A_1(e^{i\theta}, e^{i\phi})\, u_2(x, y) - \frac{q^{3/2} A_0}{(1+q)(1+q+q^2)} u_3(x, y). \tag{4.17}$$

Note that $A_1(e^{i\theta}, e^{i\phi})$ is invariant under $e^{i\theta} \to qe^{i\theta}$, hence $A_1(q^k e^{i\theta}, e^{i\phi}) = A_1(e^{i\theta}, e^{i\phi})$, $k = \pm 1, \pm 2, \ldots$. Thus $A_1(e^{i\theta}, e^{i\phi})$ appears in $\Psi_1(\cos\theta, \cos\phi \mid q)$, $\Psi_3(\cos\theta, \cos\phi \mid q), \ldots$. On the other hand $A_1(q^{1/2}e^{i\theta}, e^{i\phi})$ appears in $\Psi_2(\cos\theta, \cos\phi \mid q)$, $\Psi_4(\cos\theta, \cos\phi \mid q), \ldots$. The details will be given in Section 7.

5. A Q-ANALOGUE OF BERNOULLI POLYNOMIALS

Ismail and Zhang [18] characterized the eigenvalues and eigenfunctions of D^{-1}, on $L^2[(1-x^2)^{\nu-1/2}, -1, 1]$, where $D = \frac{d}{dx}$, and $\nu > 0$. In other words they solved the equation $D^{-1}f = \lambda f$ for

$$f \in L^2[(1-x^2)^{\nu-1/2}, -1, 1] \cap L^2[(1-x^2)^{\nu+1/2}, -1, 1], \qquad \nu > 0.$$

The operator D^{-1} was defined as an integral operator whose kernel was given as a series in ultraspherical polynomials. This is essentially the same as defining D^{-1} through its action on the ultraspherical polynomials via

$$D^{-1}C_n^{\nu+1}(x) = \frac{1}{2\nu}\, C_{n+1}^{\nu}(x). \tag{5.1}$$

In [18] it was shown that the eigenvalues are $\pm i/j_{\nu,k}$, $k = 1, 2, \ldots$, where $\{j_{\nu,k}\}$ are the positive zeros of the Bessel function of the first kind, $J_\nu(z)$, arranged so that $j_{\nu,k+1} > j_{\nu,k}$, for all k. The eigenfunctions corresponding to $\pm i/j_{\nu,k}$ are constant multiples of $\exp(\pm i j_{\nu,k}\, x)$.

Let λ_n be real for all n, $n = 0, \pm 1, \ldots$. According to Levin [19, Section 3.3] the system $\{e^{i\lambda_n x}\}$ is complete in $C[-\pi, \pi]$ if there is δ such that $\lambda_n = n - \delta_n$, $\lambda_{-n} = -n + \delta_n$, with $|n| > \delta_n > \delta > 0$, $n = \pm 1, \pm 2, \ldots$. There is no restriction on λ_0 other than being real. The sequence $\lambda_0 = 0$, $\lambda_{\pm n} = j_{\nu,\pm n}/\pi$ satisfies all the above assumptions [25], hence $\{\exp(\pm i j_{\nu,n}\, x) : n = 0, \pm 1, \ldots\}$, with $j_{\nu,0} := 0$ is complete in $C[-1, 1]$. The case $\nu = 1/2$ is interesting because $j_{1/2,n} = n\pi$ and the system $\{e^{in\pi x}\}$ is now also an orthogonal system. If $D = \frac{d}{dx}$ then its resolvent, namely $(\lambda - D)^{-1}$, is $-I_1(1-\lambda I_1)^{-1}$, which is $-\sum_{n=0}^{\infty} \lambda^n I_1^{n+1}$. Using (1.1) with n replaced by $n\pi$, and the fact $(I_1)^m = I_m$ we see that the resolvent of D is an operator whose kernel is

$$-\sum_{m=1}^{\infty} \sum_{n \neq 0}^{\infty} \frac{\lambda^m}{(in\pi)^m} \exp(in\pi(x-y)). \tag{5.2}$$

The n-sum is $-2^m B_m((x-y)/2)/m!$ [10] and the above sum becomes a generating function for Bernoulli polynomials [10]. Therefore the Green's function of D (the kernel of the resolvent operator) is

$$G(x, y; \lambda) = -1 + \frac{2\lambda}{e^{2\lambda} - 1} \exp(\lambda(x-y)). \tag{5.3}$$

Using an analysis similar to that in [18] we see that the eigenvalues of D are $in\pi$, with $n=0, \pm 1, \ldots$. Clearly the λ-singularities of $G(x, y; \lambda)$ are at $\lambda = in\pi$, $n=0, \pm 1, \ldots$ and the residue at $\lambda = in\pi$ is $e^{in\pi x}\overline{e^{in\pi y}}$ as expected.

We now come to q-Bernoulli polynomials. Define polynomials $\{u_n(x, y)\}$ by

$$u_n(\cos\theta, \cos\phi) := 2^{-n} e^{-in\theta} (q^{(1-n)/2} e^{i(\theta+\phi)}, q^{(1-n)/2} e^{i(\theta-\phi)}; q)_n. \tag{5.4}$$

It is easy to see that $u_n(x, y)$ is a polynomial in x and in y of degree n, for all $n \geqslant 0$. The first few u_n's are

$$\begin{aligned} &u_0(x, y) = 1, \qquad u_1(x, y) = x - y, \\ &u_2(x, y) = x^2 + y^2 - xy(q^{1/2} + q^{-1/2}) + \tfrac{1}{4}(q^{1/2} - q^{-1/2})^2. \end{aligned} \tag{5.5}$$

Furthermore

$$\mathscr{D}_q u_n(x, y) = \frac{1-q^n}{1-q} q^{(1-n)/2} u_{n-1}(x, y). \tag{5.6}$$

Observe that

$$u_{2n}(x, y) = \prod_{j=0}^{n-1} [x^2 + y^2 - xy(q^{-j-1/2} + q^{j+1/2}) + \tfrac{1}{4}(q^{-j-1/2} - q^{j+1/2})^2], \tag{5.7}$$

$$u_{2n+1}(x, y) = (x-y) \prod_{j=1}^{n} [x^2 + y^2 - xy(q^{-j} + q^j) + \tfrac{1}{4}(q^{-j} - q^j)^2]. \tag{5.8}$$

Clearly $u_n(x, y) \to (x-y)^n$ as $q \to 1$. Furthermore u_n has the symmetries

$$u_n(-y, -x) = u_n(x, y), \qquad u_n(y, x) = (-1)^n u_n(x, y).$$

Since $\mathscr{D}_q \Psi_m(x, y \mid q) = \Psi_{m-1}(x, y \mid q)$ and $\Psi_1(x, y \mid q)$ is given by (3.15), $x = \cos\theta$, $y = \cos\phi$, it then follows by induction that there exist functions $A_2(e^{i\theta}, e^{i\phi}), \ldots, A_n(e^{i\theta}, e^{i\phi})$ such that $\mathscr{D}_{q,x} A_j(e^{i\theta}, e^{i\phi}) = 0$, $j = 1, \ldots, n$ (q-constants) and

$$\begin{aligned} &\Psi_n(\cos\theta, \cos\phi \mid q) \\ &\quad = -\sum_{k=0}^{n} A_{n-k}(q^{k/2} e^{i\theta}, e^{i\phi}) \frac{(1-q)^k}{(q; q)_k} q^{k(k-1)/4} u_k(\cos\theta, \cos\phi), \end{aligned} \tag{5.9}$$

where $A_0(e^{i\theta}, e^{i\phi}) = 1/4$, and $A_1(e^{i\theta}, e^{i\phi})$ is as in (3.15) and (4.16). In view of the discussion at the end of Sections 4 and 9 it can also be proved inductively that all but one of the A_j's, namely A_1 are in fact absolute constants with $A_3 = A_5 = \cdots = 0$. These facts will be established in Section 7. When n is an even positive integer the coefficient of $u_{n-1}(x, y)$ is

$$\frac{(1-q)^{n-1}}{(q;q)_{n-1}} q^{(n-1)(n-2)/4} A_1(q^{1/2}e^{i\theta}, e^{i\phi}),$$

and, if n is odd the coefficient is

$$\frac{(1-q)^{n-1}}{(q;q)_{n-1}} q^{(n-1)(n-2)/4} A_1(e^{i\theta}, e^{i\phi}),$$

as observed before. The reason for the presence of $q^{1/2}$ in some of the subsequent A's is the fact that

$$(\mathscr{D}_q fg)(x) = \breve{f}(q^{1/2}e^{i\theta})(\mathscr{D}_q g)(x), \qquad \text{if} \quad \mathscr{D}_q f = 0. \tag{5.10}$$

It is clear then that (5.9) can be written in the form

$$\begin{aligned} -\Psi_n(\cos\theta, \cos\phi \mid q) &= \sum_{k=0}^{n} A_{n-k}(e^{i\theta}, e^{i\phi}) \frac{(1-q)^k}{(q;q)_k} q^{k(k-1)/4} u_k(\cos\theta, \cos\phi) \\ &\quad + [A_1(q^{(n-1)/2}e^{i\theta}, e^{i\phi}) - A_1(e^{i\theta}, e^{i\phi})] \\ &\quad \times \frac{(1-q)^{n-1}}{(q;q)_{n-1}} q^{(n-1)(n-2)/4} u_{n-1}(\cos\theta, \cos\phi). \end{aligned}$$

Thus the second term on the right-hand side in the above equation is not zero only if n is even. This and the above considerations establish the generating function

$$\begin{aligned} -\sum_{n=0}^{\infty} \Psi_n(x, y \mid q)(2t)^n &= \frac{(q^{1/2}t^2(1-q)^2; q^2)_\infty}{(q^{-1/2}t^2(1-q)^2; q^2)_\infty} \sum_{n=0}^{\infty} A_n(e^{i\theta}, e^{i\phi})(2t)^n \\ &\quad \times \mathscr{E}_q(x, -y; (1-q)\, tq^{-1/4}) \\ &\quad + t[A_1(q^{1/2}e^{i\theta}, e^{i\phi}) - A_1(e^{i\theta}, e^{i\phi})] \frac{(q^{1/2}t^2(1-q)^2; q^2)_\infty}{(q^{-1/2}t^2(1-q)^2; q^2)_\infty} \\ &\quad \times [\mathscr{E}_q(x, -y; (1-q)\, tq^{-1/4}) - \mathscr{E}_q(x, -y; -(1-q)\, tq^{-1/4})]. \end{aligned} \tag{5.11}$$

In view of [10, p. 37],

$$\frac{2^k}{k!} B_k(x/2) = -\sum_{n\neq 0} \frac{\exp(i\pi n x)}{(in\pi)^k}, \tag{5.12}$$

we may define the q-Bernoulli polynomials $\{B_n(x, y \mid q)\}$ as constant multiples of the polynomials $\Psi_n(2x, 2y \mid q)$. This definition is not suitable if we require $\mathscr{D}_q B_n(x, y)$ to be a constant multiple of $B_{n-1}(x, y)$. The reason is that $\mathscr{D}_q$ and the change of variable $x \to 2x$ do not have a simple commuting relation as in the case of the differentiation operator. This makes us settle for the following definition

$$B_n(x, y \mid q) = -(q; q)_n\, 2^n \Psi_n(x, y \mid q). \tag{5.13}$$

Therefore

$$B_0(x, y \mid q) = 1, \qquad \frac{B_1(x, y \mid q)}{(1-q)} = \frac{x-y}{2} + 2A_1(e^{i\theta}, e^{i\phi}), \tag{5.14}$$

and we have the generating function

$$\begin{aligned} -\sum_{n=0}^{\infty} B_n(x, y \mid q) \frac{t^n q^{n/4}}{(q; q)_n} &= \frac{(qt^2(1-q)^2; q^2)_\infty}{(t^2(1-q)^2; q^2)_\infty} \sum_{n=0}^{\infty} A_n(e^{i\theta}, e^{i\phi})\, q^{n/4}(2t)^n \\ &\quad \times \mathscr{E}_q(x, -y; (1-q)\, t) \\ &\quad + t[A_1(q^{1/2}e^{i\theta}, e^{i\phi}) - A_1(e^{i\theta}, e^{i\phi})]\, \frac{(qt^2(1-q)^2; q^2)_\infty}{(t^2(1-q)^2; q^2)_\infty} \\ &\quad \times [\mathscr{E}_q(x, -y; (1-q)\, t) - \mathscr{E}_q(x, -y; -(1-q)\, t)] \end{aligned} \tag{5.15}$$

Comparing the generating function (5.11) with the generating function

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{te^{xt}}{e^t - 1} \tag{5.16}$$

[10], we see that the sequence $A_0, A_1(e^{i\theta}, e^{i\phi}), \ldots$ plays the role of Bernoulli numbers.

Let $y_0 = [q^{1/4} + q^{-1/4}]/2$. Observe that in view of (3.21) and (3.22) it follows that $B_n(x, y_0)$ are essentially the Bernoulli polynomials introduced by Suslov in [22]. This can be seen from the material at the end of Section 3.

q-analogues of some of the properties of Bernoulli polynomials and numbers can be derived from the results in this section. The first example is the analogue of

$$B_n(x+z)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)\,z^k. \tag{5.17}$$

The notation $f(x\mathring{+}y)$ was used in [15] to denote $E_q^y f(x)$ $(=E_q^x f(y))$. Now (3.24), (5.15) and the fact that q-constants are invariant under q-translations imply

$$B_n(x\mathring{+}z,y\,|\,q)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q B_{n-k}(x,y\,|\,q)\,\frac{g_k(z)}{(q;q)_k}\left(\frac{1-q}{2q^{1/4}}\right)^k, \tag{5.18}$$

where we used

$$\mathscr{E}_q(x;\alpha)=\sum_{n=0}^{\infty}\frac{g_n(x)}{(q;q)_n}\alpha^n. \tag{5.19}$$

Connections with a q-analogue of the zeta function are under investigation.

An attractive approach would be to use the extra degree of freedom in $\Psi_n(x,y\,|\,q)$ by simply taking $y=-y_0$, y_0 being $(q^{1/4}+q^{-1/4})/2$. This will give a q-analogue of the identity

$$B_n(x+1)-B_n(x)=nx^{n-1}.$$

6. A KERNEL FOR $\mathscr{D}_q^{-1}$

Let $\{p_n(x;\mathbf{a})\}$ be a multiparameter family of polynomials satisfying the orthogonality relation

$$\int_E p_m(x;\mathbf{a})\,p_n(x;\mathbf{a})\,w(x;\mathbf{a})\,dx=h_n(\mathbf{a})\,\delta_{m,n}, \tag{6.1}$$

where $\mathbf{a}$ stands for the multiparameter vector $(a_1,...,a_r)$. Assume further that we have a lowering operator T so that

$$Tp_n(x;\mathbf{a})=u_n(\mathbf{a})\,p_{n-1}(x;\mathbf{a}+\mathbf{1}), \tag{6.2}$$

where $\mathbf{a}+\mathbf{1}=(1+a_1,...,1+a_r)$. When T is $\mathscr{D}_q$ then the continuous q-ultraspherical polynomials and the Askey–Wilson polynomials are examples of this set up. The Poisson kernel of $\{p_n(x;\mathbf{a})\}$ is

$$P_t(x,y;\mathbf{a})=\sum_{n=0}^{\infty}\frac{p_n(x;\mathbf{a})\,p_n(y;\mathbf{a})}{h_n(\mathbf{a})}t^n. \tag{6.3}$$

Since one would expect

$$\lim_{t\to 1^-} \int_E P_t(x, y; \mathbf{a})\, f(y)\, w(y; \mathbf{a})\, dy = f(x), \tag{6.4}$$

to hold, we defined [16] a formal right inverse to T to be T^{-1}, where

$$(T^{-t}f)(x) = \int_E K^t(x, y; \mathbf{a})\, w(y; \mathbf{a}+1)\, f(y)\, dy, \tag{6.5}$$

and the kernel $K^t(x, y; \mathbf{a})$ is given by

$$K^t(x, y; \mathbf{a}) = \sum_{n=0}^{\infty} \frac{p_{n+1}(x; \mathbf{a})}{h_n(\mathbf{a}+1)} \frac{p_n(y; \mathbf{a}+1)}{u_{n+1}(\mathbf{a})}\, t^n. \tag{6.6}$$

In this section we find explicitly a kernel K which corresponds to K^t with $t = 1$, the polynomials are the Askey–Wilson polynomials. This provides a representation of a right inverse to $\mathscr{D}_q$. A crucial step in the proof uses the concept of a q-integral, [4, 12].

The Askey–Wilson polynomials correspond to $\mathbf{a} = (a, b, c, d)$ and

$$\begin{aligned} P_n(\cos\theta; \mathbf{a}) &= P_n(\cos\theta; a, b, c, d \mid q) \\ &= {}_4\phi_3\left(\begin{matrix} q^{-n}, & abcdq^{n-1}, & ae^{i\theta}, & ae^{-i\theta} \\ ab, & ac, & ad \end{matrix} \,\middle|\, q, q\right). \end{aligned} \tag{6.7}$$

The kernel $K(x, y); \mathbf{a})$ is split as $K_1(x, y; \mathbf{a}) - K_2(x, y; \mathbf{a})$ as in the last equality in (6.17). The kernel K_1 is given by (6.19) while K_2 is given by (6.22).

In the Askey–Wilson case the u_n and h_n are defined through

$$u_{n+1}(\mathbf{a}) = \frac{2aq^{-n-1}(1-q^{n+1})(1-abcdq^n)}{(1-q)(1-ab)(1-ac)(1-ad)}, \tag{6.8}$$

and

$$\begin{aligned} \frac{1}{h_n(\mathbf{a}+1)} &= \frac{(q, abq, acq, adq, bcq, bdq, cdq; q)_\infty}{2\pi(abcdq^2; q)_\infty} \\ &\quad \times \frac{(1-abcdq^{2n+1})}{(1-abcdq)} \frac{(abcdq, abq, acq, adq; q)_n}{(q, cdq, bdq, bcq; q)_n\, a^{2n} q^n}, \end{aligned} \tag{6.9}$$

respectively. Substitute for u_n and h_n from (6.8) and (6.9) in (6.6) to find that the kernel K^t when $t=1$ becomes

$$K(\cos\theta, \cos\phi; \mathbf{a}) = \frac{(1-q)(1-ab)(1-ac)(1-ad)\,q}{2ah_0(\mathbf{a})} \times \sum_{n=0}^{\infty} \frac{1-abcdq^{2n+1}}{1-abcdq} \frac{(abcdq, abq, acq, adq; q)_n}{a^{2n}\,(q, cdq, bdq, bcq; q)_n} \times \frac{P_{n+1}(\cos\theta; \mathbf{a})\, P_n(\cos\phi; \mathbf{a}+1)}{(1-q^{n+1})(1-abcdq^n)}. \tag{6.10}$$

However, Exercise 7.34 in [12] provides a q-integral representation for the Askey–Wilson polynomials as moments of a discrete measure which takes the form

$$P_{n+1}(\cos\theta; \mathbf{a}) = \frac{1}{A(\theta)} \frac{(cd; q)_{n+1}}{(ab; q)_{n+1}} \times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcdu/q; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty} \times \frac{(q/u; q)_{n+1}}{(abcdu/q; q)_{n+1}} \left(\frac{abu}{q}\right)^{n+1} d_q u, \tag{6.11}$$

and

$$P_n(\cos\phi; \mathbf{a}+1) = \frac{1}{B(\phi)} \frac{(bdq; q)_n}{(acq; q)_n} \int_{q^{1/2}e^{i\phi}/c}^{q^{1/2}e^{-i\phi}/c} \frac{(cvq^{1/2}e^{i\phi}, cvq^{1/2}e^{-i\phi}, abcdqv; q)_\infty}{(cav, cbv, cdv; q)_\infty} \times \frac{(q/v; q)_n}{(abcdqv; q)_n} (acv)^n\, d_q v, \tag{6.12}$$

where $A(\theta)$ and $B(\phi)$ are given by

$$A(\theta) = \frac{q(1-q)}{2ib} (q, ac, ad, cd; q)_\infty \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{h(\cos\theta; a, c, d)} \csc\theta, \tag{6.13}$$

and

$$B(\phi) = \frac{q^{1/2}(1-q)}{2ic} (q, abq, adq, bdq; q)_\infty \times \frac{(e^{2i\phi}, e^{-2i\phi}; q)_\infty \csc\phi}{h(\cos\phi; aq^{1/2}, bq^{1/2}, dq^{1/2})}, \tag{6.14}$$

respectively. We then substitute for the ${}_4\phi_3$'s in (6.10) from (6.11) and (6.12) and derive the following q-double integral representation for our kernel

$$K(\cos\theta, \cos\phi; \mathbf{a}) = \frac{(1-q)(1-cd)(1-ac)(1-bc)}{2c(1-abcd)(1-abcdq)\,h_0(\mathbf{a})} \frac{1}{A(\theta)\,B(\phi)} \times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \int_{q^{1/2}e^{i\phi}/c}^{q^{1/2}e^{-i\phi}/c} \frac{(bue^{i\theta}, bue^{-i\theta}, abcdu/q; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty} \times \frac{(cvq^{1/2}e^{i\phi}, cvq^{1/2}e^{-i\phi}, abcdqv; q)_\infty}{(cav, cbv, cdv; q)_\infty} \frac{1-abcdv}{v(1-1/v)} \times \sum_{n=0}^{\infty} \frac{1-abcdq^{2(n+1)-1}}{1-abcdq^{-1}} \frac{(abcdq^{-1}, ad; q)_{n+1}}{(q, bc; q)_{n+1}} \left(\frac{bcuv}{q}\right)^{n+1} \times \frac{(q/u, 1/v; q)_{n+1}}{(abcdu/q, abcdv; q)_{n+1}}\, d_qu\, d_qv.$$

The n-sum becomes a ${}_6\phi_5$ with one term missing. Hence

$$K(\cos\theta, \cos\phi; \mathbf{a}) = -\frac{(1-q)(1-ac)(1-bc)(1-cd)}{2c(1-abcdq)(1-abcd)\,h_0(\mathbf{a})} \frac{1}{A(\theta)\,B(\phi)} \times \int_{q^{1/2}e^{i\phi}/c}^{q^{1/2}e^{-i\phi}/c} \frac{(cvq^{1/2}e^{i\phi}, cvq^{1/2}e^{-i\phi}, abcdv; q)_\infty}{(cav, cbv, cdv; q)_\infty\,(1-v)} \times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcdu/q; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty} \times \left\{ {}_6\phi_5\left(\begin{matrix} abcdq^{-1}, (abcdq)^{1/2}, -(abcdq)^{1/2}, ad, q/u, 1/v \\ (abcd/q)^{1/2}, -(abcd/q)^{1/2}, bc, abcdu/q, abcdv \end{matrix} \,\middle|\, q, \frac{bcuv}{q} \right) - 1 \right\} \times d_qv\, d_qu. \tag{6.15}$$

Now the above ${}_6\phi_5$ is very well-poised and can be summed by [12, (II.20)]. Its sum is

$$\frac{(abcd, bcu/q, bcv, abcduv/q; q)_\infty}{(bc, abcdu/q, abcdv, bcuv/q; q)_\infty}. \tag{6.16}$$

This evaluation reduces (6.15) to the form

$$
\begin{aligned}
K(x, y; \mathbf{a}) &= \frac{(1-q)(1-ac)(1-bc)(1-cd)}{2ch_0(\mathbf{a})(1-abcd)(1-abcdq)\, A(\theta)\, B(\phi)} \\
&\times \int_{q^{1/2}/e^{i\phi}/c}^{q^{1/2}e^{-i\phi}/c} \frac{(cvq^{1/2}e^{i\phi}, cvq^{1/2}e^{-i\phi}, abcdv; q)_\infty}{(cav, cbv, cdv; q)_\infty\, (1-v)} \\
&\times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcdu/q; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty}\, d_q v\, d_q u \\
&- \frac{(1-q)(1-ac)(1-cd)(abcdq^2; q)_\infty}{2ch_0(\mathbf{a})(bcq; q)_\infty} \\
&\times \int_{q^{1/2}e^{i\phi}/c}^{q^{1/2}e^{-i\phi}/c} \frac{(cvq^{1/2}e^{i\phi}, cvq^{1/2}e^{-i\phi}; q)_\infty}{(cav, cdv; q)_\infty\, (1-v)} \\
&\times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcduv/q; q)_\infty}{(bau/q, bdu/q, bcuv/q; q)_\infty}\quad d_q v\, d_q u. \\
&= K_1(x, y; \mathbf{a}) - K_2(x, y; \mathbf{a}),
\end{aligned}
\tag{6.17}
$$

say. By (2.10.18) and (2.10.19) in [12] we establish

$$
\begin{aligned}
K_1(x, y; \mathbf{a}) &= \frac{(1-ac)(1-bc)(1-cd)}{2ch_0(\mathbf{a})(1-abcd)(1-abcdq)} \frac{(abdq^{1/2}e^{-i\phi}, q^{3/2}e^{-i\phi}/c; q)_\infty}{(q^{1/2}e^{-i\phi}/c, abdq^{3/2}e^{-i\phi}; q)_\infty} \\
&\times {}_8\phi_7\left(\begin{matrix} abdq^{1/2}e^{-i\phi}, q^{5/4}(abd)^{1/2}e^{-i\phi/2}, -q^{5/4}(abd)^{1/2}\, e^{-i\phi/2}, \\ q^{1/4}(abd)^{1/2}\, e^{-i\phi/2}, -q^{1/4}(abd)^{1/2}\, e^{-i\phi/2}, \end{matrix}\right. \\
&\qquad \left.\begin{matrix} q, aq^{1/2}e^{-i\phi}, bq^{1/2}e^{-i\phi}, dq^{1/2}e^{-i\phi}, abcd \\ abdq^{1/2}e^{-i\phi}, bdq, adq, abq, q^{3/2}e^{-i\phi}/c \end{matrix}\,\middle|\, q, \frac{q^{1/2}e^{i\phi}}{c}\right) \\
&= \frac{(1-ac)(1-bc)(1-cd)(1-q)(q, aq/c, bq/c, dq/c; q)_\infty}{2ch_0(\mathbf{a})(1-abcd)(1-abcdq)(abq, adq, bdq, abdq/c; q)_\infty} \\
&\times {}_8W_7(abd/c; ab, ad, bd, q^{1/2}e^{i\phi}/c, q^{1/2}e^{-i\phi}/c \mid q, q),
\end{aligned}
\tag{6.18}
$$

where we used the transformation (III.24) in [12] in the last step. Thus, (6.18) and some manipulations lead us to the representation

$$
\begin{aligned}
&K_1(x, y; \mathbf{a}) \\
&= \frac{(1-q)(q, q, ac, bc, dc, aq/c, bq/c, dq/c; q)_\infty\, (abde^{i\phi}, abde^{-i\phi}; q)_\infty}{4\pi c\, (abcd, abd/c; q)_\infty\, (q^{1/2}e^{i\phi}/c, q^{1/2}e^{-i\phi}/c; q)_\infty} \\
&\times {}_8W_7(abd/c; ab, ad, bd, q^{1/2}e^{i\phi}/c, q^{1/2}e^{-i\phi}/c \mid q, q).
\end{aligned}
\tag{6.19}
$$

Turning to the second term on the right-hand side of (6.17) we have that it is equal to

$$\int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcduv/q; q)_\infty}{(bau/q, bdu/q, bcuv/q; q)_\infty} d_q u$$

$$= -\frac{iq(1-q)}{2b \sin\theta} \frac{(q, e^{2i\theta}, e^{-2i\theta}; q)_\infty (cdv, acv, ad; q)_\infty}{h(\cos\theta; a, d)(cve^{i\theta}, cve^{-i\theta}; q)_\infty}$$

$$= A(\theta) \frac{h(\cos\theta; c)}{(ac, ad; q)_\infty} \frac{(acv, cdv; q)_\infty}{(cve^{i\theta}, cve^{-i\theta}; q)_\infty}, \tag{6.20}$$

by [12, (2.10.18)]. Now the v-integral in the kernel $K_2(x, y; \mathbf{a})$ is given by the following expression

$$\int_{q^{1/2}e^{i\phi}/c}^{q^{1/2}e^{-i\phi}/c} \frac{(cvq^{1/2}e^{i\phi}, cvq^{1/2}e^{-i\phi}, qv; q)_\infty}{(cve^{i\theta}, cve^{-i\theta}, v; q)_\infty} d_q u$$

$$= \frac{q^{1/2}(1-q)}{2ic \sin\phi} \frac{(q, q, e^{2i\phi}, e^{-2i\phi}; q)_\infty}{(q^{1/2}e^{i\phi}/c, q^{1/2}e^{-i\phi}/c; q)_\infty}$$

$$\times \frac{(qe^{i\theta}/c, qe^{-i\theta}/c; q)_\infty}{(q^{1/2}e^{i\theta+i\phi}, q^{1/2}e^{i\theta-i\phi}, q^{1/2}e^{i\phi-i\theta}, q^{1/2}e^{-i\theta-i\phi}; q)_\infty}$$

$$= \frac{B(\phi)\, h(\cos\phi; aq^{1/2}, bq^{1/2}, dq^{1/2})(q; q)_\infty\, |(qe^{i\theta}/c; q)_\infty|^2}{(abq, adq, bdq; q)_\infty\, |(q^{1/2}e^{i\phi}/c; q)_\infty\, (q^{1/2}e^{i\theta+i\phi}, q^{1/2}e^{i\theta-i\phi}; q)_\infty|^2}. \tag{6.21}$$

In the above it is assumed that c, θ, and ϕ are real. Substitution in (6.17) gives

$$K_2(\cos\theta, \cos\phi; \mathbf{a})$$

$$= \frac{(1-q)(q, q; q)_\infty\, h(\cos\theta; c, q/c)}{4\pi c h(\cos\phi; cq^{1/2}, q^{1/2}/c)}$$

$$\times \frac{w(\cos\phi; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2})}{(q^{1/2}e^{i\theta+i\phi}, q^{1/2}e^{-i\theta-i\phi}, q^{1/2}e^{i\theta-i\phi}, q^{1/2}e^{i\phi-i\theta}; q)_\infty}$$

$$= \frac{(1-q)(q, q; q)_\infty\, h(\cos\theta; c, q/c) \sin\phi}{ch(\cos\phi; cq^{1/2}, q^{1/2}/c)\; h(\cos\phi; q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})}$$

$$\times \frac{(e^{2i\phi}, e^{-2i\phi}; q)_\infty}{4\pi\, h(\cos\phi; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2})}, \tag{6.22}$$

where w is the weight function for the Askey–Wilson polynomials, namely

$$w(\cos\theta; a_1, a_2, a_3, a_4) := \prod_{j=1}^{4} h(\cos\theta; a_j).$$

The expression in (4.4) correspond to the special case $a=q^{1/4}=-c$, $b=q^{3/4}=-d$ of K_1 and K_2. Note that for these special values the ${}_8W_7$ series in (6.19) is

$$
\begin{aligned}
&{}_8\phi_7\left(\begin{matrix} q^{3/2}, q^{7/4}, -q^{7/4}, q, -q, -q^{3/2}, -q^{1/4}e^{i\phi}, -q^{1/4}e^{-i\phi} \\ q^{3/4}, -q^{3/4}, q^{3/2}, -q^{3/2}, -q, -q^{9/4}e^{-i\phi}, -q^{9/4}e^{i\phi} \end{matrix} \,\middle|\, q, q\right) \\
&= {}_6\phi_5\left(\begin{matrix} q^{3/2}, q^{7/4}, -q^{7/4}, q, -q^{1/4}e^{i\phi}, -q^{1/4}e^{-i\phi} \\ q^{3/4}, -q^{3/4}, q^{3/2}, -q^{9/4}e^{-i\phi}, -q^{9/4}e^{i\phi} \end{matrix} \,\middle|\, q, q\right) \\
&= \frac{(q^{5/4}, q^2, -q^{5/4}e^{i\phi}, -q^{5/4}e^{-i\phi}; q)_\infty}{(q^{3/2}, -q^{9/4}e^{i\phi}, -q^{9/4}e^{-i\phi}, q; q)_\infty},
\end{aligned} \tag{6.23}
$$

by [12, (II.20)]. A bit of simplification in both (6.19) and (6.22) leads to the expression given in (4.4).

7. COMPUTATION OF $\mathscr{D}_q^{-1}$ ON SOME Q-POLYNOMIALS

In this section we evaluate the action of the Askey–Wilson operator $\mathscr{D}_{q,x}$ on $u_n(x, y)$, $A_1(e^{i\theta}, e^{i\phi})\, u_{2n}(x, y)$ and $A_1(q^{1/2}e^{i\theta}, e^{i\phi})\, u_{2n-1}(x, y)$, $n \geqslant 0$, and $x=\cos\theta$, $y=\cos\phi$. This is then used to establish the parity properties of A_n alluded to in Sections 4 and 5. Our findings are stated in Theorem 7.1. By using a corollary to Theorem 7.1 we give a proof of (5.9).

From (5.4) we see that

$$
u_n(\cos\psi, \cos\phi) = (-e^{-i\phi}/2)^n \frac{h(\cos\psi; q^{(1-n)/2}e^{i\phi})}{h(\cos\psi; q^{(n+1)/2}e^{i\phi})}. \tag{7.1}
$$

We next compute $v_n(x, y)$,

$$
v_n(x, y) := \mathscr{D}_{q,x}^{-1}\, u_n(x, y). \tag{7.2}
$$

By (4.4) we find

$$
\begin{aligned}
&v_n(x, y) \\
&= \frac{(1-q)(q, q; q)_\infty}{4\pi(-2)^n\, e^{in\phi} q^{1/4}}\, h(\cos\theta; -q^{1/4}, -q^{3/4}) \\
&\quad\times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty\, h(\cos\psi; q^{(1-n)/2}e^{i\phi}, q^{(1-n)/2}e^{-i\phi})\, d\psi}{h(\cos\psi; q^{(n+1)/2}e^{i\phi}, q^{(n+1)/2}e^{-i\phi}, -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})} \\
&\quad - \frac{(-2)^{-n}\, e^{-in\phi}(q, q : q)_\infty}{4\pi q^{1/4}(q^{1/2}, q^{3/2}; q)_\infty} \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty\, h(\cos\psi; q^{(1-n)/2}e^{i\phi})\, d\psi}{h(\cos\psi; q^{3/4}, q^{5/4}, -q^{1/4}, -q^{3/4}, q^{(n+1)/2}e^{i\phi})}.
\end{aligned} \tag{7.3}
$$

By (8.5) the integral in the first term on the right-hand side is

$$\frac{2\pi}{(1-q^{n+1})(q;q)_\infty^2}\left\{q^{n/4}e^{in\phi}\frac{(1+2q^{(2n+1)/4}y+q^{n+1/2})\,h(y;-q^{(1-2n)/4})}{h(x;-q^{1/4},-q^{3/4})\,h(y;-q^{(2n+1)/4})}\right.$$

$$\left.-\frac{q^{(2n+1)/4}e^{-in\phi}h(x;q^{-n/2}e^{i\phi})}{h(x;-q^{1/4},-q^{3/4})\,h(y;q^{(2n+1)/4}e^{i\phi})}\right\}. \tag{7.4}$$

Thus the first term in (7.3) contributes

$$\frac{1-q}{1-q^{n+1}}q^{n/2}u_{n+1}-\frac{(1-q)\,q^{(n-1)/4}}{(-2)^{n+1}\,(1-q^{n+1})}\frac{h(y;-q^{(1-2n)/4})}{h(y;-q^{(5+2n)/4})}.$$

For the second term on the right-hand side of (7.3) we first apply [12, (6.3.9)] to get

$$(1+q^{1/2})\frac{(q^{(5-2n)/4}e^{i\phi},-q^{(3-2n)/4}e^{i\phi};q^{1/2})_{2n}}{(-2)^{n+1}\,e^{in\phi}q^{1/4}(q^3,q^{1-n}e^{2i\phi};q)_n}$$

$$\times{}_8W_7(q^{-n}e^{2i\phi};q^{-(2n+1)/4}e^{i\phi},q^{-(2n+3)/4}e^{i\phi},-q^{-(2n+1)/4}e^{i\phi},$$

$$-q^{-(2n-1)/4}e^{i\phi},q^{-n};q,q^{n+3})$$

$$=\frac{(q^{1/4}+q^{-1/4})(-q^{(3-2n)/4}e^{i\phi};q^{1/2})_{2n}\,(q^2;q)_n}{(-2)^{n+1}\,e^{in\phi}(q^3;q)_n}$$

$$\times{}_4\phi_3\left(\begin{matrix}q^{-n},q^{-(2n+1)/4}e^{i\phi},q^{-(2n+3)/4}e^{i\phi},q\\ q^{-n-1},-q^{(5-2n)/4}e^{i\phi},-q^{(3-2n)/4}e^{i\phi}\end{matrix}\,\middle|\,q,q\right)$$

$$=\frac{(q^{1/4}+q^{-1/4})(-q^{-1/4}e^{-i\phi},-q^{3/4}e^{i\phi};q^{1/2})_n\,(1-q^2)}{(-2)^{n+1}\,q^{n(n-2)/4}(1-q^{n+2})}$$

$$\times\frac{(-q^{-n},-q^{(5-2n)/4}e^{-i\phi};q)_n}{(-q^{1-n},-q^{(1-2n)/4}e^{-i\phi};q)_n}$$

$$\times{}_8W_7(-q^{-n};-q^2,q^{-n},q,q^{-(2n+1)/4}e^{i\phi},q^{-(2n+1)/4}e^{-i\phi};q,q^{-1/2}).$$

In the above calculation we applied the Watson transformation [12, (III.17)] twice. We next apply [12, (III.17)] then the Watson transformation and see that the last expression

$$
\begin{aligned}
&= \frac{(1-q^2)(q^{1/4}+q^{-1/4})}{(-2)^{n+1}\,(1-q^{n+2})\,q^{n(n-2)/4}}\,(-q^{-1/4}e^{-i\phi},\,-q^{3/4}e^{i\phi};\,q^{1/2})_n \\
&\quad\times \frac{(-q^{-n},\,-q^{(5-2n)/4}e^{-i\phi};\,q)_n}{(-q^{1-n},\,-q^{(1-2n)/4}e^{-i\phi};\,q)_n} \\
&\quad\times \frac{(-q^{1-n},\,-q^{(1-2n)/4}e^{-i\phi},\,-q^{(1-2n)/4}e^{i\phi},\,-q^{3/2};\,q)_\infty}{(-q^{-n},\,-q^{(5-2n)/4}e^{-i\phi},\,-q^{(5-2n)/4}e^{i\phi},\,-q^{-1/2};\,q)_\infty} \\
&\quad\times {}_4\phi_3\left(\left.\begin{matrix} q^{-1},\,q,\,q^{-(2n+1)/4}e^{i\phi},\,q^{-(2n+1)/4}e^{-i\phi} \\ -q,\qquad q^{-n-1},\qquad -q^{3/2}\end{matrix}\,\right|\,q,\,q\right).
\end{aligned}
$$

The factors in front of the ${}_4\phi_3$ simplify to

$$
\begin{aligned}
&\frac{(1-q^2)[(q^{1/4}+q^{-1/4})/2]}{(-2)^{n+1}\,(1-q^{n+2})\,q^{n(n-2)/4}}\;\frac{(1+q^{(2-2n)/4}e^{i\phi})(1+q^{(2+2n)/4}e^{-i\phi})}{(1+q^{1/2})^2\,q^{-1/2}} \\
&\quad= \frac{(1+q)(1-q^{1/2})}{2(-2)^{n+1}\,(1-q^{n+2})}\,\frac{h(\cos\phi;-q^{(1-2n)/4})}{h(\cos\phi;-q^{(5+2n)/4})}.
\end{aligned}
\tag{7.5}
$$

Hence

$$
\begin{aligned}
\mathscr{D}_{q,x}^{-1}u_n(x,y) &= \frac{1-q}{1-q^{n+1}}\,q^{n/2}u_{n+1}(x,y) - \frac{(1-q)\,q^{(n-1)/4}}{(1-q^{n+1})(-2)^{n+1}}\,\frac{h(\cos\phi;\,q^{(1-2n)/4})}{h(\cos\phi;\,q^{(5+2n)/4})} \\
&\quad- \frac{(1+q)(1-q^{1/2})\,q^{(n+1)/4}}{(-2)^{n+2}\,(1-q^{n+2})}\,\frac{h(\cos\phi;-q^{(1-2n)/4})}{h(\cos\phi;-q^{(5+2n)/4})} \\
&\quad\times\left\{1-\frac{(1-q)[1-2q^{-(2n+1)/4}\cos\phi+q^{-n-1/2}]}{(1+q)(1+q^{3/2})(1-q^{-n-1})}\right\}.
\end{aligned}
\tag{7.6}
$$

From (5.7) and (5.8) we get

$$
u_{2m}(\cos\theta,\cos\phi) = \frac{q^{-m^2}h(\cos\theta;\,q^{1/2}e^{i\phi},\,q^{1/2}e^{-i\phi})}{2^{2m}h(\cos\theta;\,q^{m+1/2}e^{i\phi},\,q^{m+1/2}e^{-i\phi})},
\tag{7.7}
$$

$$
u_{2m+1}(\cos\theta,\cos\phi) = -\frac{q^{-m(m+1)}h(\cos\theta;\,e^{i\phi},\,qe^{-i\phi})}{2^{2m+1}e^{i\phi}h(\cos\theta;\,q^{m+1}e^{i\phi},\,q^{m+1}e^{-i\phi})}.
\tag{7.8}
$$

Define a sequence $\{V_m(x,y)\}$ via

$$
V_{2m+1}(x,y) := -\mathscr{D}_{q,x}^{-1}A_1(e^{i\theta},e^{i\phi})\,u_{2m}(x,y), \qquad m \geqslant 0,
\tag{7.9}
$$

$$
V_{2m}(x,y) := -\mathscr{D}_{q,x}^{-1}A_1(q^{1/2}e^{i\theta},e^{i\phi})\,u_{2m-1}(x,y), \qquad m > 0.
\tag{7.10}
$$

First we shall find a closed form for V_{2m+1}. From (4.16) it follows that

$$A_1(e^{i\theta}, e^{i\phi}) = -\frac{1+q^{1/2}}{8e^{i\phi}} \frac{h(\cos\theta; e^{i\phi}, qe^{-i\phi})}{h(\cos\theta; q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi})}, \tag{7.11}$$

$$A_1(q^{1/2}e^{i\theta}, e^{i\phi}) = -\frac{1+q^{1/2}}{8q^{1/2}e^{-i\phi}} \frac{h(\cos\theta; q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi})}{h(\cos\theta; e^{i\phi}, qe^{-i\phi})}. \tag{7.12}$$

Hence

$$\begin{aligned}
&-\frac{2^{2m+3}q^{m^2}e^{i\phi}}{1+q^{1/2}} e^{i\phi}V_{2m+1}(x, y) \\
&= \frac{(1-q)(q;q)_\infty^2}{4\pi q^{1/4}} h(x; -q^{1/4}, -q^{3/4}) \\
&\quad \times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty \, h(\cos\psi; e^{i\phi}, qe^{-i\phi})\, d\psi}{h(\cos\psi; q^{m+1/2}e^{i\phi}, q^{m+1/2}e^{-i\phi})} \\
&\quad \times \frac{h(\cos\psi; e^{i\phi}, qe^{-i\phi})\, d\psi}{h(\cos\psi; -q^{1/4}, -q^{3/4}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})} \\
&\quad - \frac{(q, q; q)_\infty}{4\pi q^{1/4}(q^{1/2}, q^{3/2}; q)_\infty} \\
&\quad \times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty \, h(\cos\psi; e^{i\phi}, qe^{-i\phi})\, d\psi}{h(\cos\psi; q^{m+1/2}e^{i\phi}, q^{m+1/2}e^{-i\phi}, -q^{1/4}, -q^{3/4}, q^{3/4}, q^{5/4})}.
\end{aligned} \tag{7.13}$$

The first integral on the right-hand side of (7.13) can be evaluated by setting

$$\lambda = e^{i\phi}, \quad a_1 = q^{1/2}e^{i\theta}, \quad a_2 = -q^{1/4}, \quad a = q^{m+1/2}e^{i\phi}, \quad b = q^{m+1/2}e^{-i\phi} \tag{7.14}$$

in (8.2). Then (8.5) and some simplifications show that the first integral mentioned above simplifies to

$$\begin{aligned}
&\frac{2\pi q^{-1/4}e^{i\phi}/(q, q; q)_\infty}{(1-q^{2m+1})\, h(\cos\theta; -q^{1/4}, -q^{3/4})} \left\{ \frac{h(\cos\phi; -q^{1/4}, -q^{3/4})}{h(\cos\phi; -q^{m+3/4}, -q^{m+5/4})} \right. \\
&\quad \left. + \frac{q^{m^2+m+1/2}}{1+q^{1/2}} 2^{2m+1} A_1(q^{1/2}e^{i\theta}, e^{i\phi})\, u_{2m+1}(x, y) \right\}.
\end{aligned}$$

Therefore the first term on the right-hand side of (7.13) contributes

$$\frac{(1-q)\,e^{i\phi}}{2q^{1/2}(1-q^{2m+1})}\left\{(1+2q^{1/4}y+q^{1/2})\right.$$
$$\times\prod_{k=0}^{m}[1+2q^{k+3/4}y+q^{2k+1/2}][1+2q^{k+5/4}y+q^{2k+5/2}]$$
$$\left.+2^{2m+4}\frac{q^{m^2+m+1/2}}{1+q^{1/2}}A_1(q^{1/2}e^{i\theta}, e^{i\phi})\,u_{2m+1}(x.y)\right\}. \tag{7.15}$$

In order to evaluate the second integral on the right-hand side of (7.13) it is convenient to use (8.2) to write

$$h(\cos\theta; e^{i\phi}, qe^{-i\phi})$$
$$=\frac{q^{1/4}}{2e^{i\phi}}(q^{1/2};q)_\infty^2\,[h(\cos\phi;-q^{1/4},-q^{3/4})\,h(\cos\psi;-q^{1/4},-q^{3/4})$$
$$-h(\cos\phi;q^{1/4},q^{3/4})\,h(\cos\psi;q^{1/4},q^{3/4})], \tag{7.16}$$

because the integral then reduces to three ordinary Askey–Wilson integrals (it should be observed that the presence of $q^{5/4}$ instead of $q^{1/4}$ means that we cannot apply (8.5) directly to this case). After some simplifications we finally obtain the following formula

$$V_{2m+1}(x, y)$$
$$=-\frac{(1-q)\,q^m}{1-q^{2m+1}}A_1(q^{1/2}e^{i\theta}, e^{i\phi})\,u_{2m+1}(x, y)$$
$$-\frac{(1+q^{1/2})(1-q)\,q^{-m^2-1/2}}{2^{2m+4}(1-q^{2m+1})}\frac{h(y;-q^{1/4},-q^{3/4})}{h(y;-q^{m+3/4},-q^{m+5/4})}$$
$$+\frac{(1-q)^2\,q^{-m^2}2^{-2m-5}}{(1-q^{2m+1})(1-q^{2m+2})}\left\{\frac{h(y;-q^{1/4},-q^{3/4})}{h(y;q^{m+5/4},q^{m+7/4})}\right.$$
$$-\frac{2(1-q^{2m+2})(1+q^{1/2})\,h(y;q^{1/4},q^{3/4})}{(1-q)\,h(y;-q^{m+3/4},-q^{m+5/4})}$$
$$\left.+\frac{h(y;q^{1/4},q^{3/4})}{h(y;-q^{m+5/4},-q^{m+7/4})}\right\}. \tag{7.17}$$

By a similar lengthy but straightforward calculation we also find that

$$\begin{aligned}V_{2m}(x, y) = &-\frac{(1-q)\,q^{m-1/2}}{1-q^{2m}}\,A_1(e^{i\theta}, e^{i\phi})\,u_{2m}(x, y)\\&+\frac{(1+q^{1/2})(1-q)\,q^{-m^2+m-3/4}}{2^{2m+3}(1-q^{2m})}\,\frac{h(y;-q^{1/4}, -q^{3/4})}{h(y;-q^{m+1/4}, -q^{m+3/4})}\\&+\frac{(1-q)^2\,q^{-m^2+m-3/4}2^{-2m-4}}{(1-q^{2m})(1-q^{2m+1})}\left\{\frac{h(y;-q^{1/4}, -q^{3/4})}{h(y;q^{m+3/4}, q^{m+5/4})}\right.\\&+\frac{2(1-q^{2m+1})(1+q^{1/2})\,h(y;q^{1/4}, q^{3/4})}{(1-q)\,h(y;-q^{m+1/4}, -q^{m+3/4})}\\&\left.-\frac{h(y;q^{1/4}, q^{3/4})}{h(y;-q^{m+3/4}, -q^{m+5/4})}\right\}.\end{aligned} \tag{7.18}$$

We summarize our findings in the form of a theorem.

THEOREM 7.1. *With $x=\cos\theta$, $y=\cos\phi$, the actions of $\mathscr{D}_{q,x}^{-1}$ on $u_n(x, y)$, $-A_1(e^{i\theta}, e^{i\phi})\,u_{2m}(x, y)$, and $-A_1(e^{i\theta}, e^{i\phi})\,u_{2m-1}(x, y)$, are given by* (7.6), (7.17), *and* (7.18), *respectively.*

The following corollary follows from (7.17) and (7.18).

COROLLARY 7.1. *We have*

$$\mathscr{D}_{q,x}^{-1}u_n(x, y) = \frac{(1-q)\,q^{n/2}}{1-q^{n+1}}\,u_{n+1}(x, y)+f(y), \tag{7.19}$$

$$\begin{aligned}&\mathscr{D}_{q,x}^{-1}A_1(q^{n/2}e^{i\theta}, e^{i\phi})\,u_n(x, y)\\&\quad=\frac{(1-q)\,q^{n/2}}{1-q^{n+1}}\times A_1(q^{(n+1)/2}e^{i\theta}, e^{i\phi})\,u_{n+1}(x, y)+g(y),\end{aligned} \tag{7.20}$$

where f and g depend on n and y but not on x.

Proof of (5.9). This argument will prove (5.9) and $A_{2n+1}=0$ for $n>0$. For $n=2$ formula (5.9) follows from (4.15). Assume now that (5.9) holds for $1<n\leqslant m$ and $A_{2n+1}=0$ for $2<2n+1\leqslant m$. Apply (4.1) to (5.9) with $n=m$ and make use of Corollary 7.2 to obtain

$$\begin{aligned}\Psi_{m+1}(x, y\,|\,q) = &-\sum_{k=0}^{m} A_k(q^{n+1-k)/2}e^{i\theta}, e^{i\phi})\,\frac{(1-q)^{n+1-k}}{(q;q)_{n+1-k}}\\&\times q^{(n+1-k)(n-k)/4}u_{n+1-k}(x, y)+h(y),\end{aligned} \tag{7.21}$$

where h doen not depend on x. In (7.21), interchange x and y, multiply by $(-1)^{m+1}$ and substract the result from (7.21). In view of (3.11), (5.7)–(5.8), (3.15), and (7.12) we see that $(-1)^{m+1} h(x) - h(y) = 0$. In reaching this conclusion we used the facts

$$A_{2j+1} = 0, \qquad \text{for} \quad 1 < 2j+1 \leqslant m, \qquad A_1(q^k e^{i\theta}, e^{i\phi}) = -A_1(q^k e^{i\phi}, e^{i\theta}).$$

This shows that h must be a constant A_{m+1}, say, such that $(-1)^{m+1} A_{m+1} = A_{m+1}$. Thus A_{m+1} must vanish if m is even and the proof is complete. ∎

8. APPENDIX

Let

$$S(a_1, ..., a_k) = \prod_{j=1}^{k} (a_j, q/a_j)_\infty. \tag{8.1}$$

Then, by using [12, Ex. 2.16], one can prove that

$$\begin{aligned} h(z; \lambda, q/\lambda) = {} & \frac{S(\lambda a_1, \lambda/a_1)}{S(a_2/a_1, a_1 a_2)} h(z; a_2, q/a_2) \\ & + \frac{S(\lambda a_2, \lambda/a_2)}{S(a_1/a_2, a_1 a_2)} h(z; a_1, q/a_1), \end{aligned} \tag{8.2}$$

which, on iteration, yields the identity

$$\prod_{i=1}^{r} h(z; \lambda_i, q/\lambda_i) = \sum_{j=1}^{r+1} \prod_{k=1}^{r} S(\lambda_k a_j, \lambda_k/a_j) \prod_{i=1, i \neq j}^{r+1} \frac{h(z; a_i, q/a_i)}{S(a_i/a_j, a_i a_j)}. \tag{8.3}$$

This enables us to compute the Askey–Wilson type integral

$$J := \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{h(\cos\psi; a, b)} \frac{\prod_{i=1}^{r} h(\cos\psi; \lambda_i, q/\lambda_i)}{\prod_{j=1}^{r+1} h(\cos\psi; a_j, q/a_j)} \, d\psi. \tag{8.4}$$

We just use (8.3) in (8.4) and apply the Askey–Wilson formula (6.1.1) in [12] to obtain

$$\begin{aligned} J = {} & \frac{2\pi}{(1-ab)(q; q)_\infty^2} \sum_{j=1}^{r+1} \prod_{k=1}^{r} S(\lambda_k a_j, \lambda_k/a_j) \\ & \times \prod_{i=1, i \neq j}^{r+1} [S(a_i/a_j, a_i a_j)(a a_j, b a_j, a q/a_j, b q/a_j; q)_\infty]^{-1}. \end{aligned} \tag{8.5}$$

We shall now show that (8.2) also helps us rewrite the first term of (4.11) in a form that leads to (4.13). Set $z=\cos\theta$, $a_1=q^{1/2}e^{i\phi}$, $a_2=e^{-i\phi}$, $\lambda=-q^{1/4}$ in (8.2) to get

$$\begin{aligned}
&h(\cos\theta;-q^{1/4},-q^{3/4})\\
&\quad=\frac{(-q^{3/4}e^{i\phi},-q^{-1/4}e^{-i\phi},-q^{1/4}e^{-i\phi},-q^{5/4}e^{i\phi};q)_\infty}{(q^{-1/2}e^{-2i\phi},q^{3/2}e^{2i\phi},q^{1/2},q^{1/2};q)_\infty}\\
&\qquad\times h(\cos\theta;e^{-i\phi},qe^{i\phi})\\
&\qquad+\frac{(-q^{1/4}e^{i\phi},-q^{1/4}e^{-i\phi},-q^{3/4}e^{i\phi},-q^{3/4}e^{-i\phi};q)_\infty}{(q^{1/2}e^{-2i\phi},q^{1/2}e^{2i\phi},q^{1/2},q^{1/2};q)_\infty}\\
&\qquad\times h(\cos\theta;q^{1/2}e^{-i\phi},q^{1/2}e^{i\phi}).
\end{aligned}\tag{8.6}$$

However,

$$\frac{(-q^{-1/4}e^{-i\phi},-q^{5/4}e^{i\phi};q)_\infty}{(-q^{-1/2}e^{-2i\phi},-q^{3/2}e^{2i\phi};q)_\infty}=-q^{1/4}e^{i\phi}\frac{(-q^{1/4}e^{i\phi},-q^{3/4}e^{-i\phi};q)_\infty}{(q^{1/2}e^{-2i\phi},q^{1/2}e^{2i\phi};q)_\infty},$$

and

$$\begin{aligned}
(q^{1/2}e^{-2i\phi},q^{1/2}e^{2i\phi};q)_\infty&=(q^{1/2}e^{-2i\phi},q^{1/2}e^{2i\phi},q^{3/2}e^{-2i\phi},q^{3/2}e^{2i\phi};q^2)_\infty\\
&=h(\cos\phi;q^{1/4},q^{3/4})\,h(\cos\phi;-q^{1/4},-q^{3/4}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&(q^{1/2},q^{1/2};q)_\infty\,h(\cos\theta;-q^{1/4},-q^{3/4})\,h(\cos\phi;q^{1/4},q^{3/4})\\
&\quad=h(\cos\theta;q^{1/2}e^{i\phi},q^{1/2}e^{-i\phi})-q^{1/4}e^{i\phi}h(\cos\theta;e^{-i\phi},qe^{i\phi})\\
&\quad=h(\cos\theta;q^{1/2}e^{i\phi},q^{1/2}e^{-i\phi})+2(x-y)\,q^{1/4}h(\cos\theta;qe^{i\phi},qe^{-i\phi}).
\end{aligned}\tag{8.7}$$

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